COMPUTATION OF RESPONSE ENVELOPES IN A LATTICE MATERIAL WITH SPATIOTEMPORAL MODULATIONS

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Abstract—Materials with spatiotemporal modulations possess effective properties that vary in space and time periodically. Because of the wave-like properties of a modulated material, propagation of incident waves through the material depends on their directions of travel. This direction-dependent transmission occurs due to a scattering effect that is caused because of the modulations. If the modulated material is long enough, it can act as a unidirectional wave isolator, preventing waves from propagating in one direction. However, such unidirectional transmission does not occur in very short modulated materials, i.e. two degrees of freedom (2 DOF) systems. In this work, we study nonreciprocal vibration transmission in a discrete one-dimensional (1-D) modulated material, with a focus on computing the change in the amplitude and phase of transmitted vibrations along opposite directions. Because the response of a modulated system is not periodic in time, this process requires either brute force computations or asymptotic analysis with a limited range of validity. To overcome this shortcoming, we develop and utilize the envelopes of the steady-state output displacements to investigate nonreciprocity. Furthermore, we highlight the application of envelope equations in identifying nonreciprocal response regimes characterized by a nonreciprocal phase shift in transmitted vibrations. The role of the length of 1-D modulated materials on determining the amplitude difference is highlighted. The analysis method based on envelopes of the steady-state response facilitates future parametric studies on nonreciprocity in discrete modulated materials.

Keywords—reciprocity; metamaterials; envelope equations; spatiotemporal modulation; phase nonreciprocity

I. INTRODUCTION

Reciprocity invariance is a property of regular materials with constant density and elastic modulus, functioning in the linear operating regime. When the reciprocity invariance holds, the wave propagation between two arbitrary points in the material remains unchanged after interchanging the locations of the vibration source and receiver in a material. There exist many established applications of the principle of reciprocity, for instance, calibration of hydrophones and crack identification [1,2]. Despite its usefulness, this invariance is accompanied by limitations: for example, it is impossible to send waves along a reciprocal transmission channel such that transmission properties (speed, amplitude, phase, etc.) depend on the direction of travel. In order to realize direction-dependent wave propagation, the reciprocity invariance needs to be broken. The physics and applications of nonreciprocal propagation of mechanical waves have drawn the attention of many researchers in recent decades [3].

One strategy to realize nonreciprocal wave propagation is to use a medium in which one or more of the effective properties change with time [4]. In this context, periodic materials have been often used for investigating nonreciprocal wave propagation, in the form of discrete or continuous models with modulation. The smallest repetitive sub-structure in a periodic material, known as the unit cell, determines the properties of the periodic material acting as a waveguide. Modulation is a time-varying term within an effective property of the material, usually the stiffness coefficient or elastic modulus. Unidirectional propagation was studied in an infinite-long modulated metamaterial, in which there is a wave-like spatiotemporal modulation in the elastic coefficient of the resonant spring in every discrete unit cell [5]. Nonreciprocal wave propagation can also appear in uniform continuous media after introducing spatiotemporal modulation to the elastic modulus only [6-8], or both elastic modulus and density which is known as two-phase modulation [9]. Spatiotemporal modulations are often realized in experiments with controllable external magnetic forces [10-13]. A well-established method is to generate time-varying magnetic field by tuning the current flowing through a coil, in the center of which a magnet can move along the axis of the coil. The magnet is seen as a mass in a discrete modulated system.

In contrast to the initial theoretical studies, experimental demonstrations of nonreciprocal vibration transmission due to spatiotemporal modulation were performed on setups that are necessarily finite in length [10-13]. The influence of the finite length of the system is a relevant factor when considering device implementation. At the limit of finite length, nonreciprocal vibration transmission was investigated numerically in a system with just two degrees of freedom [14].
Nonreciprocity was attributed in this system to the phase difference between the modulation properties of the two units. In this work, we investigate the influence of the length of the system (number of modulated cells) on the vibration transmission properties. Furthermore, we present a methodology for identifying nonreciprocal phase shifts in the transmitted waves. These are response regimes in which only the phase (and not the amplitude) of the transmitted wave depends on the direction of travel.

We introduce the problem formulation and solution methodology in Section 2. In Section 3, we perform a parametric study to identify the influence of system parameters on the nonreciprocal response of the system. We highlight phase nonreciprocity in Section 4. Our findings are summarized in Section 5.

II. ANALYSIS OF A COUPLED SYSTEM WITH MODULATION

We consider a system with n degrees of freedom (nDOF) which is composed of n identical masses, dampers, coupling springs and modulated grounding springs whose stiffness coefficients are time-dependent. Only the longitudinal rectilinear movement of each mass is considered as a degree of freedom. See Fig. 1.

![Diagram of nDOF system](image)

Figure 1. Scheme of the nDOF system. Stiffness coefficient of each grounding spring has two components: a constant term and a periodic term.

A. Formulation of the Problem

In the modulated system shown in Fig. 1, each coupling spring is linear and each damper is a linear viscous damper. Two external harmonic forces of the same frequency are applied on the first mass and the last mass: \( F_1 = F_1 \cos(\omega_1 t) \) and \( F_n = F_n \cos(\omega_1 t) \). Stiffness coefficient of the grounding spring connected to the \( p \)th \((p = 1, 2, \ldots, n)\) mass is: \( k_p = k_{g,DC} + k_{g,AC} \cos(\omega_m t - \phi_p) \), where \( \phi_p = (p - 1)\phi \). \( k_{g,DC}, k_{g,AC} \) and \( \phi \) are constant. The phase shift between the modulation in \( k_p \) and \( k_{p+1} \), \( \phi \), represents a spatial modulation along the length of the system. A non-zero \( \phi \) in the modulated system is the key factor in breaking the reciprocity invariance [14]. We define the following dimensionless variables to replace the dimensional terms in the governing equations: \( \tau = t \omega_0 \), \( \omega_0^2 = k_{g,DC}/m \), \( \Omega_m = \omega_m/\omega_0 \), \( \Omega_f = \omega_f/\omega_0 \), \( \zeta = c/(2m\omega_0) \), \( K_e = k_e/k_{g,DC} \), \( K_m = k_{g,AC}/k_{g,DC} \), \( P_s = F_s/(ak_{g,DC}) \), \( P_n = F_n/(ak_{g,DC}) \) and \( x_p = u_p/a \), where \( a \) is a representative length. The non-dimensionalized equations of motion for the system in Fig. 1 are:

\[
\begin{align*}
\frac{d^2}{d\tau^2} x_1 + 2\zeta \frac{dx_1}{d\tau} + x_1 + [1 + K_m \cos(\Omega_m \tau)] x_1 + K_e (x_1 - x_2) = P_1 \cos(\Omega_f \tau), \\
\vdots \\
\frac{d^2}{d\tau^2} x_p + 2\zeta \frac{dx_p}{d\tau} + x_p + [1 + K_m \cos(\Omega_m \tau - \phi_p)] x_p + K_e (2x_p - x_{p-1} - x_{p+1}) = 0, \\
\vdots \\
\frac{d^2}{d\tau^2} x_n + 2\zeta \frac{dx_n}{d\tau} + x_n + [1 + K_m \cos(\Omega_m \tau - \phi_n)] x_n + K_e (x_n - x_{n-1}) = P_n \cos(\Omega_f \tau). 
\end{align*}
\]

(1)

In this study, we focus on investigating nonreciprocity in the steady-state response of the system. In order to distinguish two opposite directions of vibration transmission, two configurations are defined: (i) the forward (left to right) configuration with \( P_1 = P_n = 0 \) where the output is the steady-state response of the last mass \( x_n^F(\tau) \); (ii) the backward (right to left) configuration with \( P_1 = 0, P_n = P \) where the output is the steady-state response of the first mass \( x_1^F(\tau) \). If and only if \( x_n^F(\tau) = x_1^B(\tau) \), the reciprocity invariance holds in vibration transmission through the system.

B. Solution Methodology and Envelopes for Outputs

Using the averaging method, we can obtain the approximated solution for the steady-state response of a modulated system [14]. The approximated solutions for outputs in forward and backward configurations read:

\[
\begin{align*}
x_n^F(\tau) &= y_n^F(\tau) e^{i \Omega_m^F \tau} + cc., \\
x_1^B(\tau) &= y_1^B(\tau) e^{i \Omega_m^B \tau} + cc.,
\end{align*}
\]

(2)

where cc. represents the corresponding complex conjugate. \( y_n^F \) and \( y_1^B \) are both functions of \( \tau \):

\[
y_n^F(\tau) = \sum_{q=-\infty}^{\infty} \xi_q e^{i q \Omega_m^F \tau}, \quad y_1^B(\tau) = \sum_{q=-\infty}^{\infty} \eta_q e^{i q \Omega_m^B \tau},
\]

where \( \xi_q \) and \( \eta_q \) are complex amplitudes for a given \( \Omega_m \), which need to be calculated. Because of the quasi-periodic form of the approximated solution in (2), \( x_n^F(\tau) \) and \( x_1^B(\tau) \) can be rewritten as:

\[
\begin{align*}
x_n^F(\tau) &= 2|y_n^F(\tau)| \cos(\Omega_f \tau - \theta_n^F), \\
x_1^B(\tau) &= 2|y_1^B(\tau)| \cos(\Omega_f \tau - \theta_1^B).
\end{align*}
\]

(3)

The cosine parts in equations of (3) have same frequency as the external excitation. These terms can be viewed as carrier waves; their amplitudes are both equal to unity but their phases can be different. Therefore, \( \pm 2|y_n^F(\tau)| \) and \( \pm 2|y_1^B(\tau)| \) are the envelopes of \( x_n^F(\tau) \) and \( x_1^B(\tau) \), respectively. The expressions for \( 2|y_n^F(\tau)| \) and \( 2|y_1^B(\tau)| \) can be written as:

\[
\begin{align*}
2|y_n^F(\tau)| &= \sum_{r=0}^{\infty} Y_{n,r}^F \cos(r \Omega_m^F \tau - \theta_{n,r}^F), \\
2|y_1^B(\tau)| &= \sum_{r=0}^{\infty} Y_{1,r}^B \cos(r \Omega_m^B \tau - \theta_{1,r}^B),
\end{align*}
\]
where $r$ is an integer. $Y_{nr}^f$, $\theta_{nr}^f$, $Y_{nr}^b$, and $\theta_{nr}^b$ are all real numbers, which can be calculated from $\xi_n$ and $\eta_n$. While $x_n^f(\tau)$ and $x_n^b(\tau)$ are not periodic, $2|y_n^f(\tau)|$ and $2|y_n^b(\tau)|$ are periodic with the same dimensionless period $T_{ev} = 2\pi/\Omega_n$. This periodicity of envelope equations brings convenience for analyzing $x_n^f(\tau)$ and $x_n^b(\tau)$. The necessary and sufficient condition for reciprocity can be written in the following two equations: $|y_n^f(\tau)| = |y_n^b(\tau)|$ and $\phi_n^f = \phi_n^b$.

Fig. 2 shows the displacements and envelopes calculated for (1) for the following parameters: $K_c = 0.6$, $\zeta = 0.01$, $K_m = 0.1$, $\Omega_m = 0.2$, $\phi = 0.5\pi$ and $P = 1$, in the steady-state with respect to time $\tau$. In order to validate the predictions made by the averaging method, the response of (1) is computed using the Runge-Kutta method until the steady state is reached. The predictions of the steady-state response from analytical and numerical methods match with each other very well. Plots of $\pm 2|y_n^f(\tau)|$ and $\pm 2|y_n^b(\tau)|$ follow the envelopes of outputs in forward and backward configurations, respectively. Having validated the accuracy of the envelope equations, results from the averaging method are used hereafter to analyze the steady-state response and nonreciprocity of the system.

Figure 2. Comparison between the results of averaging method and direct numerical simulation. (a): $n = 5$, $\Omega = 0.97$ in forward configuration; (b): $n = 8$, $\Omega = 1.18$ in backward configuration. Outputs calculated with the averaging method and the Runge-Kutta method are indicated by red curves and cyan dashed curves, respectively. The green curves are the plots of $\pm 2|y_n^f(\tau)|$ and $\pm 2|y_n^b(\tau)|$.

Because $x_n^f(\tau)$ and $x_n^b(\tau)$ are non-periodic, it is very difficult to obtain their maximum displacements. The maximum displacements of $2|y_n^f(\tau)|$ and $2|y_n^b(\tau)|$ over a period, denoted as $A_{ev,n}^f$ and $A_{ev,1}^b$, respectively, can be seen as approximations of the maximum steady-state displacements of $x_n^f(\tau)$ and $x_n^b(\tau)$. $A_{ev,n}^f$ and $A_{ev,1}^b$ can be approximated by:

$$A_{ev,n}^f = A_{ev,n,DC}^f + A_{ev,n,AC}^f, A_{ev,1}^b = A_{ev,1,DC}^b + A_{ev,1,AC}^b,$$

where,$$
A_{ev,n,DC}^f = \int_0^{T_{ev}} |y_n^f(\tau)|d\tau, A_{ev,1,DC}^b = \int_0^{T_{ev}} |y_n^b(\tau)|d\tau, \quad A_{ev,n,AC}^f = \frac{1}{T_{ev}} \int_0^{T_{ev}} [2|y_n^f(\tau)| - A_{ev,n,DC}^f]^2 d\tau, \quad A_{ev,1,AC}^b = \frac{1}{T_{ev}} \int_0^{T_{ev}} [2|y_n^b(\tau)| - A_{ev,1,DC}^b]^2 d\tau.$$

The amplitude bias $R_A$ is defined to quantify the degree of nonreciprocity in terms of amplitude, without considering the difference in phases:

$$R_A = \frac{A_{ev,n}^f - A_{ev,1}^b}{A_{ev,n}^f + A_{ev,1}^b}. \quad (4)$$

A zero amplitude bias, $R_A = 0$, corresponds to equal amplitudes for the forward and backward configurations and the limit of amplitude bias, $R_A = \pm 1$, indicates that the amplitude in one of the configurations is much larger than the other one. Furthermore, we can use $R_A = 0$ to identify response regimes where the forward and backward configurations have equal amplitudes but (possibly) different phases – see Section IV.

### III. NONRECIPROCAL VIBRATION TRANSMISSION IN MODULATED nDOF SYSTEMS

We first investigate nonreciprocity by exploring the effects of $\phi$ and $\Omega$ on maximum displacements of the outputs in the forward and backward configurations. In this section, we use $K_c = 0.6$, $\zeta = 0.01$, $K_m = 0.1$, $\Omega_m = 0.2$ and $P = 1$ and calculate $R_A$ as a function of $\phi$ and $\Omega$. The $n$ natural frequencies of the unmodulated system ($k_{g,AC} = 0$) are denoted by $\Omega_{0,k}, k = 1, 2, \ldots, n$. 

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Fig. 3 shows the amplitude bias, $R_A$, for systems with different degrees of freedom. In all four panels, regions where the magnitude of $R_A$ is largest occur where $\Omega_f$ is nearly centered at $\Omega_{0,k} \pm \Omega_m$, $\Omega_{0,k} \pm 2\Omega_m$, $\Omega_{0,k} \pm 3\Omega_m$, ...; these are called sideband resonances. As $n$ increases, the number of sideband frequencies increases and the plot of $R_A$ becomes more and more complex.

Generally, when $\Omega_f$ is fixed at a value where $R_A$ changes significantly with $\phi$, there are $2(n-1)$ convex and concave regions in the plot of $R_A$ over the range $\phi \in [0,2\pi]$. But this finding is not valid for a large $n$, for example, $n = 8$ as shown by Fig. 3d. Interestingly, all the three-dimensional (3-D) plots of $R_A$ in Fig. 3 are odd-symmetric about the line ($\phi, R_A$) = $(\pi, 0)$.

In Figs. 3 and 4, we observe that the magnitude of $R_A$ increases with the number of units in the system. Larger difference between the amplitudes of outputs can be therefore realized by adding more units into the modulated system.

Fig. 4 also shows the limiting behavior of $R_A$ for large values of $n$ in the frequency range $0.7 \leq \Omega_f \leq 2.1$. Regardless of the number of units, however, we have $-1 \leq R_A \leq 1$ by constructions, where the limiting values indicate unidirectional vibration transmission. It implies that there exist two extreme cases: $R_A = 1$ when $A_{ev,n}^F \gg A_{ev,1}^B$, which means vibrations can be transmitted in the forward direction only; $R_A = -1$ when $A_{ev,n}^F \ll A_{ev,1}^B$, which means vibrations can be transmitted in the backward direction only.

The magnitude of amplitude bias, $|R_A|$, can nearly reach 1, as shown in Fig. 4. Unidirectional vibration transmission can therefore occur in large modulated systems. This agrees with the literature on infinitely-long spatiotemporal modulated materials [5,6]. In contrast, the magnitude of $R_A$ is not very large in short modulated systems (Fig. 3). This prevents unidirectional vibration transmission from occurring in short systems. Fig. 4 suggests two ranges of values for the phase difference $\phi$ where unidirectional transmission may occur: near $0.3\pi$ and $1.7\pi$.

IV. PHASE NONRECIPROCITY

For all plots in Fig. 3, $R_A$ is equal to zero when $\phi = \pi$, regardless of the values of $\Omega_f$ and $n$. However, because $R_A$ is defined based on the envelope equations, it is blind to the phase difference between $x_0^F(\tau)$ and $x_0^B(\tau)$. Therefore, $R_A = 0$ may not correspond to a reciprocal response in the original system.
To quantify the degree of nonreciprocity in the response of the original system, we use reciprocity bias $R$ to evaluate nonreciprocity between $x_n^f(\tau)$ and $x_1^B(\tau)$:

$$R = \lim_{T \to \infty} \frac{1}{T} \int_0^T [x_n^f(\tau) - x_1^B(\tau)]^2 d\tau,$$

which is evaluated after the response reaches its steady-state [14]. If $R = 0$, then $x_n^f(\tau) = x_1^B(\tau)$ and the response is reciprocal; otherwise, the response is not reciprocal. $R$ is calculated using the averaging method with the same parameters as the examples in Fig. 3.

Figure 5. $R$ with respect to $\phi$ and $\Omega_f$. (a): $n = 2$, (b): $n = 3$, (c): $n = 5$ and (d): $n = 8$.

Fig. 5 shows 3-D plots of the reciprocity bias, $R$, as a function of $\phi$ and $\Omega_f$ for systems of different length. We observe that the 3-D plot of $R$ is symmetric about the plane $\phi = \pi$. Interestingly, if $n$ is an odd number, regardless of the value of $\Omega_f$, reciprocity invariance holds when $\phi = \pi$, as shown by Figs. 5b and 5c. In contrast, if $n$ is an even number, the reciprocity invariance does not hold when $\phi = \pi$, as shown by Figs. 5a and 5d.

The results in Figs. 3 and 5 indicate the possibility of choosing system parameters such that $R_A = 0$ and $R > 0$. In this state, $R_A = 0$ means that the amplitudes of the output in the forward and backward configurations are the same. Therefore, nonreciprocity ($R > 0$) manifests as different phases in the output displacements. The existence of such regimes of nonreciprocal phase shifts was previously reported in time-independent nonlinear systems [15], but not in modulated systems of the type considered in this work.

Figure 6. Outputs and their envelopes, (a) and (b): $n = 4$, $\Omega_f = 1.17$, (c) and (d): $n = 8$, $\Omega_f = 1.51$. 

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Fig. 6 shows two examples of phase nonreciprocity obtained at $\phi = \pi$. It is clear from the time-domain response that the response is nonreciprocal. Notice, however, that the difference between the displacements is only in a phase shift. This can be seen more clearly in the response envelopes. For any arbitrary value of $\Omega_f$, when $\phi = \pi$ and $n$ is equal to an even number, besides $R_A = 0$ and $R > 0$, we have $|y_0^\phi(\tau)| = |y_0^\beta(\tau \pm T_{ev}/2)|$. The envelopes in different configurations have the same shape with a temporal shift of $\tau$ by half period. However, $x_n^\phi(\tau)$ is not equal to $x_n^\beta(\tau + T_{ev}/2)$ or $x_n^\beta(\tau - T_{ev}/2)$ due to the phase difference between two carrier waves. Here, outputs in forward and backward configurations follow the same envelope profile but at different phases.

V. CONCLUSION

By investigating the envelopes of outputs in forward and backward configurations, we studied nonreciprocal vibration transmission in discrete models of modulated materials. Specifically, we developed equations for the envelopes of the output displacements for this purpose. While the response of modulated systems is quasi-periodic in general, the envelope equations are periodic in time. Thus, studying the envelopes of the nonperiodic response brings convenience in approximating the maximum displacements in the steady-state. We provided a measure for quantifying the degree of nonreciprocity based on the envelope of the response, the amplitude bias. We used the amplitude bias to identify two response regimes in the system. First, we showed that the maximum magnitudes of the amplitude bias correspond to unidirectional vibration transmission in the modulated system. These maximum values were obtained only in systems with many modulated units (long systems). Second, we demonstrated that zero amplitude bias can be used to identify phase nonreciprocity in the system; i.e. response regimes where the difference between the forward and backward output displacements is in their relative phase only. Our results demonstrate that the envelope equations can provide information about nonreciprocity in modulated materials that would be difficult to obtain otherwise. We observed that amplitude bias is not significant in shorter models (with fewer degrees of freedom), and it can become more significant with increasing the number of degrees of freedom. Unidirectional transmission and bandgaps were highlighted in the scenarios with very long models, which agree with the findings of directional bandgaps in infinite-long spatiotemporal modulated systems. Comparing the results of amplitude bias and reciprocity bias for different examples, we identified nonreciprocal response regimes in which the amplitude bias is zero. Furthermore, the envelopes of forward output and backward output can have the same shape and a half-period offset from each other. We presented the conditions leading to this specific form of nonreciprocity.

The analysis based on the envelope of nonperiodic steady-state response provides a new strategy for studying nonreciprocal vibration transmission in linear modulated materials. The methodology presented in this work facilitates parametric studies in the future.

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