Hybridization of a Polynomial Adaptive High-Order Solver for Unsteady Compressible Flows

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Abstract—We present $p$-adaptive hybridized flux reconstruction schemes to reduce the computational cost of implicit discretizations using a nondimensional vorticity indicator. At each adaptation level, we apply a projection operation to the new space based on the element-wise projected solution and the transmission conditions. We validate our implementation and analyze performance via numerical examples. Specifically, we show that $p$-adaptation of hybridizable flux reconstruction (HFR) methods results in comparable numerical error to standard $p$-adaptive and $p$-uniform standard flux reconstruction (FR) discretizations with a fraction of degrees of freedom. Performance results for a cylinder at $Re = 150$ showcase speedup factors in excess of 6 for hybridized methods in comparison with $p$-adaptive standard FR schemes, and up to 40 compared to $p$-uniform FR discretizations. Similarly, results for a NACA 0012 airfoil at $Re = 10,000$ demonstrate speedup factors close to 6 compared to $p$-adaptive FR discretizations, and up to 33 compared to $p$-uniform conventional FR. Hence, combining hybridization with adaptation yields a significant reduction in computational cost compared to conventional implicit discretizations.

Keywords-component—polynomial adaptation; flux reconstruction schemes; hybridization

I. INTRODUCTION

Engineering flows typically entail a range of complex structures that can be challenging to study numerically. High-order methods have been shown to be excellent candidates for problems with significant complexity due to their reduced computational cost per degree of freedom. Of particular interest is the Flux Reconstruction (FR) method introduced by Huynh for advection [1] and diffusion [2] type of problems. FR uses an element-wise polynomial representation of the solution and is able to recover existing formulations via a choice of correction functions. These include the discontinuous Galerkin (DG) [3], [4], spectral volume (SV) [5], and many others.

For problems with high numerical stiffness, such as high-Reynolds number wall-bounded flows, implicit time stepping may be preferred due to its stability properties as opposed to the conditional stability of explicit time stepping. However, for high-order discretizations, the computational cost of solving the nonlinear systems of equations that arise with implicit solver scales with $(p + 1)^d$ s for a given solution polynomial of degree $p$ and problem dimension $d$. This scaling can be reduced by one dimension via hybridization. Cockburn et al. [6], presented the hybridization to discontinuous Galerkin methods (HDG), which consists of including an additional variable into the problem, the so-called trace. This variable lives on the skeleton of the computational grid and allows decoupling internal degrees of freedom between neighboring elements. This results in a set of Dirichlet subproblems augmented with transmission conditions. Hence, we can apply static condensation to reduce the size of the implicit system, the size of which depends on the number of trace points instead. Then, we only need to solve a reduced global system and a set of local element-wise problems. The scaling of the hybridized global system is $(p + 1)^{d-1}$. The hybridization approach has been recently extended to the family of Flux Reconstruction schemes by Pereira et al. [7]. Specifically, the hybridized flux reconstruction (HFR) method and embedded flux reconstruction (EFR) method have been shown to significantly speed up numerical computations for a series of conservation laws.

Problems within the FR framework are locally high-order, and conservation is achieved weakly via numerical fluxes. This allows a natural application of local $p$-
adaptation algorithms, which enable the use of variable degrees of the solution polynomial from one element to another. Hence, it is possible to obtain a reduction in computational cost of high-order simulations while maintaining accuracy, which is suitable for problems with confined regions of large gradients.

Previous work on p-adaptive hybridizable methods [8]–[10] has focused on the HDG method where the trace variable is allowed to be globally discontinuous. In our previous work [7], we have shown that the EFR method is in fact competitive in terms of efficiency. The objective of this paper is to introduce p-adaption in the context of hybridized flux reconstruction schemes and to compare their performance against p-uniform and p-adaptive standard FR discretizations. We present a formulation that generalizes the adaptation algorithm to both discontinuous and continuous hybridized FR methods, which can be applied at intervals during a simulation while having little impact on overall computational cost.

II. THE HYBRIDIZED FLUX RECONSTRUCTION SCHEME

Consider the following conservation law

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F}(u) = 0 \text{ in } \Omega,$$

(1)

where $\Omega$ is a bounded subset of $\mathbb{R}^d$ with boundary $\partial \Omega \in \mathbb{R}^{d-1}$ and $d$ dimensions, $u$ is the conserved quantity, $\mathbf{F} = \mathbf{F}(u)$ is the flux, and $t$ is time.

In order to discretize this equation, we begin by considering a computational domain defined by a partition of nonoverlapping, conforming elements $\Omega_k$ that form the set $\mathcal{T}_h$. Each of these elements contains a boundary $\partial \Omega_k = \{ f \}$ with $|\partial \Omega_k| = N_f$. The collection of all these element boundaries are defined by $\partial \mathcal{T}_h = \{ \partial \Omega_k : \Omega_k \in \mathcal{T}_h \}$, where for two interior elements, overlapping faces coexist in this set. The intersection of all element faces in the domain defines the skeleton of the grid $\varepsilon_h = \varepsilon^0 \cup \varepsilon^0 = \tilde{f}$, where $\varepsilon^0$ refers to the boundary faces, and $\varepsilon^0$ represents the interior faces. A relationship between an element’s face and its global position in the computational grid is given by $\tilde{f} \in \varepsilon_h$, $\tilde{f} = (f \in \partial \Omega_k) \cap \varepsilon_h$.

We begin by mapping each element $\Omega_k$. The relationship between the reference and physical space is given via invertible one-to-one mapping functions $\mathbf{M}_k(\tilde{x})$. The Jacobian matrix of these transformations is denoted by $J_k(\tilde{x})$ and its determinant by $\text{det } J_k(\tilde{x})$. After application of these geometric transformations, the physical solution within each element satisfies

$$\frac{\partial u^h}{\partial t} + \frac{1}{J_k} \tilde{\nabla} \cdot \tilde{\mathbf{F}}^h_k = 0,$$

(2)

where $\tilde{\nabla}$ is the reference space divergence operator.

Within each element, we place $N_s$ interior solution points $\{ \tilde{x}_s^k \}_{s=1}^{N_s}$, that define globally discontinuous polynomials of degree $p$. The conserved variables are then represented via interpolation with nodal basis functions $\varphi(\tilde{x})$

$$\hat{u}_k^h(\tilde{x}, t) = \sum_{i=1}^{N_s} \hat{U}_{k,i}(t) \varphi_i(\tilde{x}).$$

The same interpolation can also be used to construct flux polynomials $\mathbf{F}_k^h(\tilde{x}, t)$ in the same polynomial space of the solution and is also allowed to be discontinuous at the interfaces. This flux is then corrected to have $C^0$ continuity by adding the term

$$\hat{F}_k^h(\tilde{x}, t) = \sum_{f=1}^{N_f} \sum_{m=1}^{N_{m,f}} g_{n,m}^h(\tilde{x}) \left[ \tilde{H}(\tilde{x}) \right]_{\tilde{x}=\tilde{x}^f,m},$$

(4)

where several new variables have been introduced. Among these are the correction functions $g_{n,m}$, which belong to a Raviart-Thomas space and are defined on each of the flux points $\{ \tilde{x}^f,m \}_{f=1}^{N_f}$ at each face $f$. In addition, we have introduced the flux interface jump, defined as

$$\tilde{H}_{k,f}(\tilde{x}) = \frac{z}{\tilde{g}_{k,f}} \cdot \hat{n}_f - \hat{F}_k^h \cdot \hat{n}_f,$$

(5)

where $\tilde{g}_{k,f}$ is the transformed discontinuous flux at the interface, $\hat{n}_f$ is the reference outward normal vector function, and $\tilde{g}_{k,f}$ is the hybridized Riemann flux of the form

$$\tilde{g}_{k,f} = \mathbf{F}(\hat{u}_k^h) + s(\hat{u}_k^h - \hat{u}_k^h) \hat{n}_{k,f},$$

(6)

with $\hat{n}_{k,f}$ the physical outward unit normal vector and $s$ a stabilization parameter [7], [11]. In contrast to standard FR formulations, where neighboring element information is used in the numerical flux, we have here introduced the so-called trace variable. A degree-$p$ polynomial can be constructed on each of the computational faces $f$

$$\tilde{u}_k^h(\tilde{x}, t) = \sum_{i=1}^{N_s} \hat{U}_{k,i}(t) \phi_i(\tilde{x}),$$

(7)

which is interpolated via basis functions $\{ \phi(\tilde{x}) \}$ defined in an appropriate finite-element space such as

$$\mathcal{M}_p = \{ \mu \in L_2(\varepsilon_h) : \mu | f \in \mathbb{P}^p(\tilde{f}), \forall \tilde{f} \in \varepsilon_h \},$$

(8a)

$$\tilde{M}_p = \mathcal{M}_p \cap C^0(\varepsilon_h),$$

(8b)

which respectively result in the HFR and EFR schemes. While the second space in (8) is $C^0$-continuous over the computational skeleton, the interior element solution remains globally discontinuous for all methods. To close the system, the problem statement requires an additional equation, which is known as the transmission conditions, and can be written

$$\sum_{f \in \varepsilon_h} \int_f \| \tilde{\mathbf{g}} \| \phi ds = \sum_{f \in \varepsilon_h} \int_f \tilde{g}_{BC} \phi ds = 0,$$

(9)
which implicitly enforces flux conservation. Here $\tilde{F}^{BC}$ is the normal boundary flux and $[\mathbf{f}]$ is the interface normal jump operator. Summing over all elements, the hybridized flux reconstruction takes the form

$$
\sum_{\Omega_k \in \mathcal{T}_h} \frac{\partial \tilde{u}_k^h}{\partial t} + \sum_{i=1}^{N_f} \mathbf{F}_{k,i} \cdot \nabla \psi_i(\mathbf{x}) + \sum_{f=1}^{N_f} \sum_{m=1}^{N_r} \tilde{\mathbf{n}} \cdot g^m_f(\mathbf{x}) \left[ \tilde{H}(\mathbf{x})_{k,f} \right]_{\mathbf{x} = \mathbf{x}_{\partial f,m}} = 0,
$$

(10a)

$$
\sum_{f \in \mathcal{T}_h} \int_{\partial f} \left[ \mathbf{\tilde{F}} \right] \mathbf{j} d\mathbf{s} + \sum_{f \in \mathcal{T}_h} \int_{\partial f} \mathbf{\tilde{F}}^{BC} \mathbf{j} d\mathbf{s} = 0.
$$

(10b)

This can be written after linearization for the $n$-th Newton iteration

$$
\begin{bmatrix}
A^n & B^n \\
C^n & D^n
\end{bmatrix}
\begin{bmatrix}
\delta \mathbf{u}^n \\
\delta \mathbf{\tilde{u}}^n
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{r}^n \\
\mathbf{s}^n
\end{bmatrix},
$$

(11)

where $\delta \mathbf{u}^n$ and $\delta \mathbf{\tilde{u}}^n$ refer to the update vector of internal and trace solution points at this iteration. Due to the discontinuous nature of the interior solution and decoupling neighboring elements’ interior solutions, $A$ is block-diagonal. This proves efficient when we reduce the problem via static condensation and solve the condensed global problem only in terms of the trace variable, i.e.

$$
\mathbf{L}^n \delta \mathbf{\tilde{u}}^n = \mathbf{t}^n,
$$

(12)

where $\mathbf{L} = \mathbf{D} - \mathbf{C} \mathbf{A}^{-1} \mathbf{B}$ and $\mathbf{t} = \mathbf{s} - \mathbf{C} \mathbf{A}^{-1} \mathbf{r}$. Then, the solution can be obtained locally for each element from

$$
\delta \mathbf{u}_k = \mathbf{A}_k^{-1} \left( \mathbf{r}_k - \mathbf{B}_k \delta \mathbf{\tilde{u}}_k \right).
$$

(13)

III. ADAPTATION

In practice, only regions of flow complexity, such as unsteady vortex shedding, require high resolution. This poses a problem for $p$-uniform computations since the overall computational cost is increased in regions where little flow complexity is observed, such as in the far field. The local nature of flux reconstruction schemes allows a natural implementation of adaptivity algorithms. In adaptation, we increase or decrease the degrees of elements depending on a form of error estimation. Feature-based indicators have been shown to be useful for unsteady flows. In particular, the nondimensional vorticity indicator of [12] has shown potential for these types of flow. It consists of determining the local element-wise maximum nondimensional vorticity magnitude, i.e.

$$
\theta_k = \frac{\omega_{k,l_k}}{\nu_\infty},
$$

(14)

where $\omega_k$ is the maximum vorticity magnitude

$$
\omega_k = \max_{1 \leq i \leq N_r} |\omega_{k,i}|,
$$

(15)

$l_k$ is the maximum length between two mapping points in the element, and $\nu_\infty$ is the freestream velocity. Then, the new polynomial degree is decided based on a maximum polynomial degree bound $p_{\text{max}}$, a threshold vector $c = [c_1, \ldots, c_{p_{\text{max}}}]$ [12], and a tolerance factor $\epsilon$. Depending on whether the new polynomial is higher or lower, an element-wise interpolation or $L_2$ projection of the interior solution is respectively applied on the interior solution. For hybridized methods, the updated trace can be found in line with the transmission conditions by ensuring

$$
\sum_{f \in \mathcal{T}_h} \int_{\partial f} \left[ \mathbf{\tilde{F}}(\mathbf{u}_h, \mathbf{\tilde{u}}_h) \right] \mathbf{j} \mathbf{\phi}_f d\mathbf{f} = 0,
$$

(16)

where $\mathbf{u}_h$ is the locally projected interior solution and $\mathbf{\phi}_f$ is the new trace basis function at face $f$. The degree of the trace nodal basis function after element adaptation is determined as the maximum polynomial degree between the two neighboring elements at a given interface. For the hybridized method with discontinuous traces, (16) yields a problem local to each face, which can be solved efficiently.

IV. NUMERICAL EXAMPLES

In this section, we present two numerical examples to showcase the benefits of $p$-adaptation in hybridized FR schemes. We will discuss results for flow over a cylinder at $\text{Re} = 150$ and flow over a NACA0012 airfoil at $\text{Re} = 10,000$. All of the performance computations correspond to serial runs.

A. Cylinder at $\text{Re} = 150$

In our first numerical example, we simulate flow over a cylinder at $\text{Re} = 150$. We consider the flow with $M = 0.1$. We make use of a mesh of 3564 quadrangles refined at the wall and growing away from it. A second-order two-stage SDIRK scheme was used to advance the solution in time, with a $\Delta t/t_c = 6.25 \times 10^{-3}$, which corresponds to about 100 times the maximum stable value of an explicit RK4 method using $p = 5$ for this problem. This yields a CFL number $\approx 8$. Here, $t_c = t u_\infty/D$ is the convective time. The vorticity indicator tolerance was set to $\epsilon = 0.1$ and the implicit tolerance was set to $10^{-5}$ for all cases. After transient effects of the simulation passed, flow statistics were averaged over $200 t_c$ and compared against the works of [13], [14].

Fig. (1) shows the polynomial distribution at an instantaneous snapshot corresponding to a moment of minimum lift for the adaptive FR, HFR, and EFR discretizations. It is expected that regions with higher vorticity magnitude, such as the boundary layers, will require higher resolution. The overall distribution of the adapted polynomials is similar between FR, HFR, and EFR. P4 elements can be seen in the boundary layer and cores of the unsteady vortices. The polynomial degree is gradually reduced to P3 in the vicinity of P4 and so on until P1. The latter is used mostly in the far-field regions. Right from the vicinity
of the back of the cylinder to about 15 characteristic lengths downstream, the three considered schemes agree well in the distribution of polynomial degrees for most elements. For the downstream cells beyond this point and in some regions upstream of the cylinder, EFR flagged higher values of vorticity and hence introduced higher polynomial degrees. This can be associated with higher solution jumps, resulting in spurious local vorticity values. This results in EFR having slightly more degrees of freedom than HFR and FR, as shown in Table (I).

Figure 1. Instant snapshots of the cylinder problem showing the adaptation algorithm for FR, HFR and EFR at moments of minimum lift. Contour lines of vorticity are superimposed.

We compare our numerical results against the reference data of [13] and the references therein. Specifically, we compare the averaged drag coefficient $c_d$, the amplitudes of the $c_l$ and $c_d$ curves, and the Strouhal number $St$, the latter three have been obtained via an FFT signal decomposition. From Table (I), it is clear that $p = 1$ is significantly less accurate than any higher polynomial degree. Overall, for the $p$-uniform simulations, results start to converge beyond $p = 3$ for FR, HFR, and EFR, showing little change after this value. Similar values are obtained among the three schemes. The adaptive simulations also agree well with the reference data with a relative error smaller than 0.5% in all quantities with respect to the $p = 5$ simulations. This was done at a fraction of the degrees of freedom, which we show in the last two columns of this table. The resulting internal degrees of freedom $\text{DOF}_v$ and the averaged trace degrees of freedom $\text{DOF}_e$ are presented. HFR and EFR formulations required only about a 1/2 and 1/3 of the degrees of freedom of the $p$-adaptive FR scheme in the implicit system, respectively.

Table I. Averaged results for the cylinder problem for FR, HFR and EFR schemes including both $p$-uniform and $p$-adaptive runs

<table>
<thead>
<tr>
<th>$p$</th>
<th>$c_d$</th>
<th>$\Delta c_d$</th>
<th>$\Delta c_l$</th>
<th>$St$</th>
<th>$\text{DOF}_v$</th>
<th>$\text{DOF}_e$</th>
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<tr>
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<td>0.0526</td>
<td>0.5155</td>
<td>0.1842</td>
<td>32076</td>
<td>-</td>
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<td>0.5163</td>
<td>0.1843</td>
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<td>0.0526</td>
<td>0.5161</td>
<td>0.1843</td>
<td>89100</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
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<td>0.0527</td>
<td>0.5163</td>
<td>0.1843</td>
<td>128304</td>
<td>-</td>
</tr>
<tr>
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<td>0.0527</td>
<td>0.5163</td>
<td>0.1843</td>
<td>128304</td>
<td>-</td>
</tr>
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<td>0.5186</td>
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<td>40669</td>
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</table>

A table with speedup factors is shown in Table (II), where we have computed $t_{p\text{FR}}/t_{p\text{HFR}}$ and $t_{p\text{FR}}/t_{p\text{EFR}}$, and $p_{\text{FR}}$, $p_{\text{HFR}}$ and $p_{\text{EFR}}$ are the polynomial degrees of a given FR, HFR and EFR scheme. Consistent with previous observations [7], hybridized methods show increased performance benefits at higher polynomial degrees. However, by looking at the diagonals of these tables, where we compare equal polynomial degrees for the hybridized and standard formulations, we see that at least 1.26 faster simulations were achieved at $p = 1$. As the order is increased, speedups of up to 14.5 are observed for the EFR method at $p = 5$ for $p$-uniform discretizations of the same degree. Hybridized adaptive methods achieved speedups of 5.34 and 6.42 for the HFR and EFR methods with respect to the adaptive standard FR scheme. Furthermore, we see the benefits of using $p$-adaptive hybridization, which allows us to obtain results comparable to $p = 4$ to $p = 5$ simulations by reducing the computational costs by between 15 to 45 times.

B. NACA 0012 Airfoil at $Re=10,000$

Finally, we present flow over a NACA0012 airfoil at $Re = 10,000$ with an angle of attack $\alpha = 2$ degrees and chord $c$. In this simulation, unsteady vortex shedding from the airfoil is compared between hybridized and standard FR formulations. We run this problem at a Mach number $M = 0.2$. The computational grid is structured and is
Table II. Speedup factors for the cylinder problem. Results compare runtime ratios between the FR method and the corresponding scheme for all considered $p$-adaptive and $p$-uniform runs.

<table>
<thead>
<tr>
<th>$p$</th>
<th>FR</th>
<th>HFR</th>
<th>EFR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.26</td>
<td>5.93</td>
<td>20.42</td>
</tr>
<tr>
<td>2</td>
<td>0.54</td>
<td>2.55</td>
<td>8.79</td>
</tr>
<tr>
<td>3</td>
<td>0.28</td>
<td>1.34</td>
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<tr>
<td>4</td>
<td>0.16</td>
<td>0.76</td>
<td>2.63</td>
</tr>
<tr>
<td>5</td>
<td>0.11</td>
<td>0.50</td>
<td>1.72</td>
</tr>
</tbody>
</table>

composed of 8658 quadrilateral elements, with refinement towards the airfoil walls to capture the formation of the boundary layer. The downstream length of the domain is 100c. We employ the second-order two-stage SDIRK method for time stepping with a $\Delta t = 1 \times 10^{-3}t_c$ in this problem, which gives a CFL number of approximately 2.5. This numerical example is run with uniform solution polynomial degrees $p = 1$ to $p = 5$ as well as with $p$-adaptation using FR, HFR, and EFR. For the adaptation algorithm, we use a tolerance for the vorticity indicator of $\epsilon = 0.1$. A convergence tolerance of $10^{-5}$ is used for the implicit unsteady density residual.

Fig. (2) displays the distribution of polynomial degrees over the domain for each of the considered schemes. Here we show instantaneous snapshots at moments of minimum lift. The algorithm set P5 elements in proximity to the airfoil leading edge at around $x/c \leq 0.15$ and gradually decreases the polynomial degree to P4 and then to P3 over the rest of the airfoil. Vortices shedding off the airfoil are tracked by the adaptation algorithm with P4 elements close to their core, which also gradually decreases all the way to P1 cells in the far field. The three considered schemes have comparable polynomial distributions, which result in similar numbers of degrees of freedom, as observed in Table (III).

V. CONCLUSIONS

In this work, we have presented $p$-adaptive hybridizable flux reconstruction schemes that are able to reduce the computational cost of implicit high-order discretizations. We presented a procedure for locally adapting the polynomial degrees in each cell by solving a global projection problem, which determines the new value of the trace in our functionally nonconforming setting. Due to the discontinuous nature of the HFR method, this problem is, in fact, local but remains global for EFR. Since adaptation is not performed at every iteration, the cost of this adaptation procedure for EFR is akin to that of a time step. We made use of a nondimensional vorticity indicator in our adaptation implementation. We ran two
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REFERENCES


Table III. Results for the airfoil problem for FR, HFR and EFR methods with p-uniform and p-adaptive discretizations

| p | | | | | |
|---|---|---|---|---|
| 1 | 0.0402 | 0.0430 | 0.0179 | 2.4187 | 34632 |
| 2 | 0.0409 | 0.0431 | 0.0171 | 2.4691 | 79222 |
| 3 | 0.0409 | 0.0436 | 0.0170 | 2.4677 | 138528 |
| 4 | 0.0409 | 0.0439 | 0.0170 | 2.4691 | 216450 |
| 5 | 0.0409 | 0.0438 | 0.0170 | 2.4667 | 311688 |
| 1-5 | 0.0409 | 0.0438 | 0.0170 | 2.4679 | 80368 |

Table IV. Speedup factors for the airfoil problem. Results compare runtime ratios between the FR method and the corresponding scheme for all considered p-adaptive and p-uniform runs

<table>
<thead>
<tr>
<th>PR</th>
<th>p</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
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<td>4.05</td>
<td>11.74</td>
<td>30.92</td>
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<td>4.43</td>
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numerical examples to showcase our results, namely, flow over a cylinder at Re = 150 and flow over a NACA 0012 airfoil at Re = 10, 000. Adaptive hybridized methods are able to achieve comparable errors to p-adaptive standard formulations at a fraction of the cost. We observed speedup factors up to 6 compared to standard FR p-adaptive simulations and up to 40 against p-uniform FR runs. Future work includes the application of the adaptive procedures to three-dimensional simulations as well as a comparison with other adaptivity indicators.

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