Autonomous Guidance & Control of Earth-Orbiting Formation Flying Spacecraft

Guidage et commande autonomes pour le vol en formation d'engins spatiaux en orbite terrestre

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"I'm sorry, Dave, I'm afraid I can't do that."

– HAL 9000, 2001: A Space Odyssey (1968)

One of the first references to the challenge of designing successful autonomous spacecraft.
Formation flying of spacecraft has gained a lot of interest within the engineering and scientific community in recent years. However, formation flying leads to an increased complexity of the guidance and control system, whose complexity grows rapidly with the number of spacecraft in the formation. Moreover, there is an increasing need for autonomy to decrease the cost of ground support since ground support operations are often a non-negligible part of the cost of a mission. Therefore, a formation flying guidance and control system needs to perform autonomous decisions and trade-offs in real-time to decrease the number of tasks that need to be performed by the ground segment and make formation flying affordable.

This work presents the development of analytical formation flying guidance and control laws for autonomous on-board applications. Firstly, an analytical model of relative motion for elliptical and perturbed reference orbits is developed. This model is solely based on the initial orbit elements of the reference trajectory and can predict the relative motion of any spacecraft orbiting close to the reference trajectory, taking into account the secular drift caused by the $J_2$ perturbation. Secondly, a new tool, the Fuel-Equivalent Space, is presented. The Fuel-Equivalent Space theory maps the relative orbit elements into a mathematical space where similar displacements on any axis is similar in terms of maneuvering fuel cost, therefore translating the minimum fuel problem into a simple distance minimization problem. Then, a neighbouring optimum feedback control law is developed. This feedback control law makes use of the optimal control theory to yield a semi-analytical controller that guarantees near-optimal maneuvering for any of the spacecraft orbiting close to the reference trajectory. Finally, it is shown that all these three new developments can be tied in together with simple analytical guidance laws to yield a fully autonomous guidance and control algorithm applicable to formation reconfiguration.
RÉSUMÉ

Le vol en formation d'engins spatiaux présente de nombreux avantages, tels la possibilité de reconfigurer la formation en orbite et une robustesse accrue à la défaillance de certains systèmes. Cependant, le vol en formation entraîne une complexification des systèmes de guidage et de commande. Comme les systèmes autonomes sont souvent nécessaires afin de réduire la dépendance envers le support au sol et réduire le coût d'opération des missions, les systèmes de guidage et de commande de vol en formation doivent pouvoir effectuer eux-mêmes des compromis entre plusieurs spécifications de mission souvent conflictuelles, pour ainsi rendre ce genre de mission plus abordable.

Ce document présente le développement d'algorithmes analytiques de guidage et de commande pour les applications embarquées et autonomes de vol en formation d'engins spatiaux. Premièrement, un modèle analytique de mouvement relatif pour une orbite de référence perturbée et excentrique est présenté. Ce modèle ne requiert que les éléments orbitaux initiaux de la trajectoire de référence afin de prédir le mouvement relatif naturel des satellites évoluant à proximité de cette trajectoire de référence, en tenant compte de l'effet à long terme des perturbations causées par l'aplatissement de la Terre. Ensuite, l'espace de consommation équivalente est présenté. Cet espace transpose les éléments orbitaux relatifs dans un espace mathématique où un déplacement équivalent dans n'importe quelle direction implique un coût équivalent en carburant, transformant ainsi le problème d'optimisation de la consommation en un problème géométrique de minimisation de distance. Puis, un algorithme de commande en boucle fermée quasi-optimal basé sur la théorie de la commande optimale est décrit. Cet algorithme de forme semi-analytique garantit la quasi-optimalité des manœuvres pour tout satellite évoluant en proximité de la trajectoire de référence. Finalement, il est démontré que ces trois développements peuvent être reliés ensemble par l'intermédiaire de lois de guidage simples et ainsi former une boucle de guidage et de commande entièrement autonome pour le vol en formation d'engins spatiaux en orbite terrestre.
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CHAPTER 1

Introduction

Formation flying of spacecraft caused a paradigm shift in the space mission design community in recent years. Formation flying spacecraft are spacecraft that orbit close from one another and for which the most stringent requirements are defined in terms of relative position and velocity, as opposed to spacecraft constellations which are made of several spacecraft independently guided and controlled and between which there is no interaction in terms of guidance and control (e.g. GPS satellites). Formation flying allows the replacement of large expensive spacecraft with a formation of smaller and cheaper spacecraft. Such a mission design can indeed present numerous financial and operational advantages. For example, formation flying missions can potentially have a lower production cost due to economics of scale, in the case where a single large and complex satellite is replaced by several “mass production” smaller spacecraft. Secondly, using a constellation of spacecraft could decrease the cost of launch. Launching several smaller elements is potentially cheaper than launching a single big and heavy satellite, mainly because small satellites can be launched piggy-backed on a larger spacecraft flight support equipment.

Moreover, spacecraft formation flying presents several operational advantages. The most important one is an increased robustness through failure recovery and graceful degradation. In deep-space missions using multiple spacecraft in formation, if a sub-system failure occurs in a spacecraft, another fully functional spacecraft could support the disabled spacecraft. The capabilities can be shared. For example, when a power, communication or navigation system failure occurs in a spacecraft, it may be possible to use another spacecraft sub-system either by physically linking the spacecraft or by transmitting navigation information to the failed spacecraft. In the case of a distributed spacecraft interferometer or a distributed antenna mission, the failure of one
spacecraft would only cause a “graceful degradation” of the system, rather than compromising the whole mission. Thus, failure recovery and graceful degradation of the system decrease the risk to the mission. The second operational advantage is a mission restructuring capability. It is foreseeable to reconfigure the satellite formation on-orbit to follow new mission requirements. Moreover, if the mission has multiple objectives, resources can be optimized by dispatching a certain group of spacecraft with special attributes to achieve one objective, and then command another group of spacecraft to achieve another objective in parallel.

This chapter thus presents the main challenges to be faced in the design of autonomous formation flying guidance and control and lays the foundation of the theory upon which it is built. Sections 1.1 and 1.2 give an overview of the future of formation flying and what it is likely to mean in terms of guidance and control requirements. Section 1.3 presents the main top-level architectures that have been applied to formation flying. Section 1.4 presents the main aspects of formation flying guidance. Section 1.5 presents the various ways of modeling formation flying spacecraft relative motion. Section 1.6 describes the formation flying control strategies available in the literature. Finally, section 1.7 presents the main objectives of the research project and section 1.8 describes the structure of the chapters of the thesis in relationship with those project objectives.

1.1 Future Missions and the Need for Increased Autonomy

One of the most interesting aspect of formation flying is that it could lead to innovative space systems applications which could not be foreseen with conventional monolithic satellites and with a large dependency on ground support. Examples of these applications are the distributed radio-frequency antenna, the swarm of optical elements, the on-orbit formation acquisition and the automated on-orbit assembly [5, 56].
1.1. FUTURE MISSIONS AND THE NEED FOR INCREASED AUTONOMY

Figure 1.1 Swarm of Radio-Frequency Elements

**Distributed Radio-Frequency Antenna** The distributed radio-frequency antenna can be used to send/receive data to/from a remote location, replacing large monolithic reflector apertures (Figure 1.1). In this case, each spacecraft carries a transponder that regenerates the sensed signal, but with a time delay corresponding to the relative position of the spacecraft. Thus, the receiver gets an amplified coherent signal. This can be achieved with either loosely distributed elements or with a stable formation such as a circular satellite formation.

If the number of spacecraft is very large, this concept can become a fully distributed system (Figure 1.2). It could instantaneously create transmitter and receiver elements within the swarm and provide a multi-spotting capability. This system then becomes a very robust and flexible communication system.

**Swarm of Optical Elements** A swarm of spacecraft carrying optical elements can be used to replace large telescopes. In this scenario, several small reflectors are distributed and reflect the incoming signal to a receiver (Figure 1.3). Because it has no structure, this design can be substantially lighter than a filled reflector. This system becomes a zooming telescope if the spacecraft have a reconfiguration capability.

**On-Orbit Formation Acquisition** In a formation acquisition scenario (Figure 1.4), all the spacecraft can be integrated as one single unit during launch and transfer. When the carrier vehicle has reached the target location, the spacecraft sequentially leave the carrier to reach the specified formation pattern. For example, in
CHAPTER 1. INTRODUCTION

Figure 1.2 Distributed Radio-Frequency Antenna

Figure 1.3 Swarm of Optical Elements
a Mars observation mission, the spacecraft could be launched within a vehicle carrier that ensures the transfer between the Earth and Mars. When the carrier reaches the vicinity of Mars, the spacecraft are launched and place themselves in the adequate formation for observation. Once the mission is complete, the spacecraft come back and dock to the carrier vehicle. The carrier could bring the spacecraft to another orbit or to another planet. This "mother ship" approach has also been considered for manned exploration missions [5]. It could provide significant savings in terms of resources for life support. It could also have interesting benefits for Earth-orbiting missions if, for example, on-orbit refuelling is needed or if the orbit of the formation has to be changed.

**On-Orbit Automatic Assembly** A swarm of small spacecraft could perform on-orbit assembly of a large space structure, such as an on-orbit station, a deep-space com-
munication antenna or a radio-telescope. Some of the spacecraft could even become part of the final structure and share their capabilities. The same technology can be used to reconfigure or disassemble a space-based structure (Figure 1.5).

However, all these missions require autonomous guidance and control, mainly for two reasons. First, such missions involve too many complex maneuvers to be commanded from the ground. The cost of ground support would be prohibitive. Second, the communication delay between the Earth and the spacecraft would make the generation of low-level ground commands impractical. When the spacecraft evolve in a rapidly evolving environment, such as during maneuvers or when several spacecraft are in close vicinity, the reaction time of the guidance and control system becomes critical. Such a reaction time would be affected if ground support is in the control loop. This need for autonomy inevitably adds constraints in the design of the guidance and control system.

1.2 Guidance & Control Challenging Requirements

Obviously, using a formation of spacecraft involves several challenges. The first one is an increase of the required level of autonomy, as described earlier. In order to mini-
mize the resources needed for ground support, it is required to limit the system command inputs to high-level commands to the whole swarm of spacecraft. The swarm of spacecraft would then have to autonomously define lower-level commands for every member of the formation. The second challenge is the design of a fuel-optimal control system. Formation keeping should maximize the lifetime of the whole formation with fuel-optimal strategies. Furthermore, if the consumption of fuel is not well balanced among the members of the formation, some spacecraft could run out of fuel before other ones, and cause a premature degradation of the performance of the system.

Guidance and control has to autonomously manage several conflicting requirements: mission requirements, formation propellant lifetime, single spacecraft propellant lifetime and collision avoidance. Mission requirements usually consist of a desired configuration and a desired formation accuracy. Formation propellant lifetime requirements necessitate that the fuel consumption of the whole swarm is globally minimized to maximize the lifetime of the formation. Single spacecraft propellant lifetime requirements put constraints on the fuel consumption difference between elements of the formation so that all the elements of the formation have an almost identical life cycle duration. Finally, collision avoidance has also to be taken into account, especially during reconfiguration maneuvers, for obvious reasons.

In all cases, the most fuel-efficient way of dealing with the requirements is to use natural motion of the spacecraft to perform maneuvers instead of fighting natural motion. In a scenario where the spacecraft use electric propulsion, the order of magnitude of the force generated by the actuators is the same as the order of magnitude of the perturbation forces. Therefore, it is profitable to use those perturbations to perform the maneuvers instead of fighting the perturbations to remain on a trajectory that does not take the perturbations into account. However, to be able to use the natural motion, it is essential to accurately model it with a suitable relative motion theory.
1.3 Formation Flying Guidance & Control Architectures

The guidance and control architecture defines the way information is shared between spacecraft and how the reference trajectories are generated. In Ref. [41], the authors divide the formation flying guidance and control architectures into five categories: Multiple-Input Multiple-Output, Leader/Follower, Virtual Structure, Cyclic and Behavioral. This classification is used here to present the different formation flying architectures that can be found in the literature.

**Multiple-Input Multiple-Output** In the Multiple-Input Multiple-Output (MIMO) architecture, the relative motion between the spacecraft is controlled by considering each spacecraft as elements of a system to control. This is a highly centralized approach where all the information needs to be at the same computational node. The main advantage of MIMO algorithms is optimality and stability. The fuel consumption and the accuracy of the formation can be optimized for all the spacecraft and the stability of the whole system can be verified. However, this type of architecture is not robust to local failures. The failure of one system of one of the spacecraft could make the whole formation unstable. A local failure could potentially have a global effect.

**Leader/Follower** The Leader/Follower architecture is by far the most studied architecture. It uses a hierarchical structure that reduces the problem to a set of individual tracking problems. Each spacecraft of the formation is given a reference to track. The reference point to track can be either another spacecraft, a virtual point on orbit or a set of virtual states. Most of the formation control algorithms have been developed in this framework, mainly because the relative motion models are well suited to study Leader/Follower type formation flying.

**Virtual Structure** A Virtual Structure is a set of rigidly connected virtual nodes that provide reference states for each of the spacecraft in the formation at any point in time. The main advantage of the Virtual Structure is that it ensures the formation
is maintained during maneuvers. It is also an efficient collision-avoidance algorithm. The Virtual Structure approach has been mainly applied to deep-space formation flying [39, 23, 3]. For Earth-orbiting formations, the Virtual Structure approach would prove to be poorly fuel-efficient. Tracking a "rigid" formation is most of the time not the natural trajectory of the spacecraft.

**Cyclic** The cyclic architecture, as described in Ref. [41], is an interconnection of individual spacecraft controllers that result in a cyclic control dependency. Therefore, each spacecraft is at the same time a leader and a follower. This control architecture is particularly suitable for circular formations where the dependency between the spacecraft can be clearly established. However, the stability of this kind of formation has only been verified through simulation (as opposed to a more rigorous theoretical demonstration).

**Behavioral** The behavioral architecture combines multiple inputs from competing controllers to achieve conflicting goals. The control action is based on a weighted average of control strategies to achieve each objective. Typical formation flying behaviours include collision avoidance, goal seeking and formation keeping. The control laws associated with each of the behaviours are usually based on the same theory as the Leader/Follower control laws. Up to now, the behaviour-based control architecture has only been applied to deep-space formation flying [28, 41].

Most of the theoretical work on formation flying has been performed for the Leader/Follower case. In this case there is a fixed reference to track. Whatever is the number of spacecraft in the formation, the problem is reduced to individual tracking problems. In turn, the MIMO approach optimizes the maneuvers of all the spacecraft at the same time. However, the robustness of the control algorithm to system failures and the amount of information that has to be shared make it unsuitable for large autonomous and decentralized formations.

The Virtual Structure architecture, the cyclic architecture and the behavioral architecture would all require control algorithms such as the ones that have been developed
for the Leader/Follower architecture. These methods mainly differ in the way the reference trajectory is defined.

The MIMO, the Leader/Follower and the Virtual Structure architectures are considered as centralized approaches because the reference states for all the spacecraft are coming from one "omniscient" source. The cyclic and the behavioral architectures are considered to be decentralized approaches, because each spacecraft makes its own decisions based on its perception of the environment. No single entity requires the full knowledge of the states and the intentions of all the elements of the formation.

For the remainder of the current work, the Leader/Follower approach shall be considered, mainly because solutions developed for the Leader/Follower architecture are likely to be applicable to the other types of architectures. Once the reference trajectory and/or target formations are known, Leader/Follower solutions usually apply.

1.4 Formation Flying Guidance

As stated earlier, guidance and control of autonomous formation flying face many conflicting requirements. It is mainly the role of the guidance algorithm through the generation of reference trajectories to perform trade-offs between these requirements. However, trade-offs will be different depending on whether the formation is in a maintenance or a reconfiguration mode.

Maintenance is defined as maintaining a desired relative formation. The objectives of the guidance and control algorithms at this time are mainly to maintain the formation within the accuracy tolerances while spending as little propellant as possible. Reconfiguration is defined as going from an arbitrary initial formation to a given desired configuration. Reconfiguration could be fuel-optimal and/or time-optimal depending on the mission requirements or the type of the spacecraft propulsion system. Maintenance and reconfiguration modes both lead to different challenges.
1.4.1 Formation Maintenance Challenges

For the Keplerian unperturbed case, non-drifting relative motion can be ensured only if the orbital energy of both spacecraft, characterized by the orbit semi-major axis, is identical. This causes both spacecraft to have the same orbital period. Thus, relative motion will be exactly periodic over one orbit. However, many perturbations are encountered around the Earth. The most important one is that caused by the oblateness of the Earth, commonly referred to as $J_2$ perturbation. This non-sphericity of Earth's gravitational field has many impacts on the orbital dynamics that can be classified in three categories (Fig. 1.6):

- Nodal regression
- Apsidal line rotation
- Libration (orbit or half-orbit periodic oscillations)

![Diagram of J2 perturbation effects on orbit elements]

Figure 1.6 Effect of the $J_2$ Perturbation on the Orbit Elements

When the $J_2$ perturbation is considered, two different spacecraft, with two different set of orbit elements but with the same orbital energy, could experience different orbit
element drift caused by the $J_2$ perturbation. Over time, this will result in a drift of the relative motion. Depending on the orbit elements of the spacecraft, this could have a non-negligible effect over tens of orbits.

It is not possible to design orbits that would experience no secular drift of their orbit elements. However, it is possible to design $J_2$-invariant relative orbits [44, 46] that would not experience any secular relative drift of their orbit elements. In other words, both spacecraft orbit elements would drift, but at the same rate, so that the relative motion would not drift over time.

The perturbations encountered in Earth’s gravitational field cause three types of perturbations: short-period oscillations, long-period oscillations and secular drift. Using Brouwer’s theory [8], it is possible to extract mean orbit elements, from which short-term oscillations have been removed. The mean orbit elements, as opposed to the osculating or instantaneous orbit elements, only show secular drift and long-term oscillations. Their dynamics can be described analytically with very simple expressions. With a first-order approximation, Schaub [44, 46] uses this result and defines $J_2$-invariance conditions for relative orbits. These conditions are two linear constraints on the selection of the orbit semi-major axis $a$, the eccentricity $e$ and the inclination $i$ (the reader is referred to Appendix A for a more thorough definition of the orbit elements used in this work). In other words, two constraints restrict the choice of three of the six orbit elements. Mathematically speaking, the $J_2$-invariant set in the $\{a, e, i\}$ mean orbit element space is simply defined by a straight line. The choice of the three other orbit elements (the right ascension of the ascending node $\Omega$, the argument of perigee $\omega$, and the mean anomaly $M$) has no impact on the relative secular drift of the relative orbits. All these conclusions are only valid in the mean orbit element space, so that osculating orbit elements (or “actual” orbit elements) will still show a short-period oscillation of their relative motion. However, osculating orbit elements will not show any long-term relative drift (to a first-order approximation).

In any cases, formation lifetime and accuracy are improved if formations are designed using $J_2$-invariant relative orbits [44]. The use of mean orbit elements can lead to spe-
1.4. FORMATION FLYING GUIDANCE

cific conditions under which, to a first-order approximation, two spacecraft would not experience any long-term secular drift. Only short-period oscillations would be observed in the relative motion between spacecraft. These oscillations, if undesirable, have to be controlled through relative orbit elements control techniques.

1.4.2 Formation Reconfiguration Challenges

The consideration of the $J_2$ perturbation is also important when time comes to perform maneuvers. This is particularly relevant with electrical low-thrust actuators. $J_2$-induced acceleration perturbations are typically of the order of $10^{-2} \text{ m/s}^2$. When considering that the current technology of electrical propulsion generates forces typically in the order of $10^{-3} \text{ N}$, and because this perturbation is so predictable, it becomes obvious that this perturbation cannot be fought but must instead be used. The easiest way to include the $J_2$ perturbation in the computation of maneuvers is of course to use a model of the relative motion that includes and predicts $J_2$ effects. This can be achieved by the development of an analytical albeit accurate model of relative motion.

Even with chemical propulsion systems, that can generate forces of several orders of magnitude larger, the $J_2$ perturbation has to be considered. Reconfiguration maneuvers are most likely to be made over one or two orbits [43]. If the $J_2$ perturbation is not accounted for, the formation accuracy will be limited by the drifting effect of $J_2$ during the number of orbits required to fully perform the maneuvers.

Collision avoidance is also an issue during reconfiguration maneuvers. In autonomous systems, the most common approach for collision avoidance is the use of potential functions [31, 36]. In this case, a cost is added for having two elements of the formation too close from one another, just as if they were two particles electrically charged with similar sign. This will naturally cause the elements of the formation to stay sufficiently far from each other. Singh and Hadaegh [50] suggest instead to consider an exclusion sphere in the computation of the reconfiguration optimal path planning. However, this method requires up-front numerical optimization from the ground, and the complexity of the problem rapidly grows with the number of spacecraft. Finally, another
method that solves the collision avoidance issue is the use of Virtual Structures (Section 1.3). However, as stated earlier, the resulting "rigid" formation would be far from the "natural" trajectory of the members of the formation, which would make this solution costly in propellant consumption.

Therefore, autonomous guidance system design meets very different challenges in the reconfiguration and the maintenance case. These challenges are also greatly impacted by the type of propulsion system and the tightness in the tolerance of the formation accuracy. However, in all cases, the main challenges are to autonomously perform trade-offs between several conflicting requirements.

1.4.3 Known Solutions

The most common solutions to formation flying guidance make use of computationally expensive techniques. Such examples are linear programming [51, 35], multi-agent optimization techniques [57], particle swarm optimization [24], genetic algorithms [2] or optimal control theory [9, 55]. These techniques use highly powerful numerical optimization algorithms to solve for the best maneuver to perform to reach a desired formation. They intrinsically have much freedom in the quantity that is to be optimized. However, in most cases, convergence is not guaranteed and the number of required numerical iterations typically cannot be predicted. As a result, these kind of techniques are not suited for on-board implementation.

On the other hand, analytical solutions to the optimal reconfiguration problem can be found, but only under certain conditions. Unperturbed circular reference orbits lead to simple analytical expressions and easily expressed configurations [40] which, in turn, pave the way for analytical solutions. For example, Mishne [33] almost analytically solves the optimal control problem for circular orbits for power-limited thrusters (only a small amount of numerical optimization remains). Furthermore, Vaddi et al. [53] developed an analytical and simple solution to the circular formation establishment and reconfiguration using impulsive thrusters about a circular reference orbit. On the other hand, Gurfil [14] proposes an analytical and optimal way of reaching bounded rela-
tive motion for any Keplerian orbits with only one impulse through the application of an energy-matching constraint. However, even though this impulse guarantees orbit-periodic relative motion, it is not made to aim for a specific configuration. Therefore, a set of analytical tools for the computation of the optimal reconfiguration maneuver about elliptical orbits for any arbitrary formation is yet to be developed. Obviously, such tools would first have to encompass a suitable relative motion theory.

1.5 Relative Motion Theories

Autonomous guidance and control systems require accurate but simple models of reality. Models have to be accurate enough to prevent unnecessary fuel expenditure and simple enough to allow implementation on a typical computational power-limited space-qualified on-board computer. If perturbation models are included in the on-board model of reality, natural motion induced by these perturbations can be used to perform the maneuvers. If these perturbations are not included, the guidance and control system will most likely compensate for the perturbations, therefore leading to unnecessary fuel expenditure.

All the relative motion models can be classified based on the assumptions they use. The simplest models assume the spacecraft orbit close to a circular reference orbit, while some others assume an elliptical reference orbit. For both cases, some models consider the reference orbit is unperturbed (perfect Keplerian motion) while others take into account the most important orbital perturbations in their prediction of relative motion.

1.5.1 Unperturbed Circular Reference Orbits

The most widely used relative orbital motion model is the Clohessy-Wiltshire-Hill (CWH) model [54, 40]. This model provides a time-explicit closed-form analytical solution to the relative motion problem for circular unperturbed orbits. This model predicts the relative motion in a Local-Vertical Local-Horizontal (LVLH) frame of a deputy with respect to a chief, orbiting on a reference circular unperturbed orbit around a spherical
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body (figure 1.7). In the LVLH frame, the $x$ axis is aligned with the chief's position vector, $z$ is normal to the orbital plane (in the direction of orbital angular momentum) and $y$ completes the right-hand frame ($y$ is aligned with the orbital velocity for circular orbits).

![Figure 1.7 CWH Model Frame](image-url)

Under these assumptions, the linearized equations of the relative motion of the deputy assuming a small distance between the chief and the deputy are (see Ref. [54] for a detailed derivation of this model):

\[
\begin{align*}
\ddot{x} - 2ny - 3n^2x &= f_x \\
\dot{y} + 2nx &= f_y \\
\ddot{z} + 2n^2z &= f_z
\end{align*}
\]

where $\rho = \begin{bmatrix} x & y & z \end{bmatrix}^T$ is the position of the deputy in the chief-centered LVLH frame (position often referred to as "Hill coordinates"), $n$ is the mean orbital motion (or angular velocity) of the reference orbit and $f_x, f_y, f_z$ are the perturbation or control accelerations in $x, y$ and $z$. Assuming an unperturbed and uncontrolled motion, a closed-form solution for the relative position and velocity as a function of the elapsed
time \( t \) can be obtained:

\[
x(t) = \frac{\dot{x}_0}{n} \sin(nt) - \left( 3x_0 + \frac{2\dot{y}_0}{n} \right) \cos(nt) + \left( 4x_0 + \frac{2\dot{y}_0}{n} \right) \\
y(t) = \left( 6x_0 + \frac{4\dot{y}_0}{n} \right) \sin(nt) + \frac{2\dot{x}_0}{n} \cos(nt) - (6nx_0 + 3\dot{y}_0) t + \left( y_0 - \frac{2\dot{x}_0}{n} \right) \\
z(t) = z_0 \cos(nt) + \frac{\dot{z}_0}{n} \sin(nt) \\
\dot{x}(t) = \dot{x}_0 \cos(nt) + (3nx_0 + 2\dot{y}_0) \sin(nt) \\
\dot{y}(t) = (6nx_0 + 4\dot{y}_0) \cos(nt) - 2x_0 \sin(nt) - (6nx_0 + 3\dot{y}_0) \\
\dot{z}(t) = -z_0 n \sin(nt) + \dot{z}_0 \cos(nt)
\]

where \( x_0, y_0, z_0, \dot{x}_0, \dot{y}_0 \) and \( \dot{z}_0 \) are the components of the initial position and velocity of the spacecraft in the LVLH frame. Lovell and Tragesser [30] reparametrized this model and demonstrated that the in-plane and out-of-plane, non-drifting, relative motion about a circular unperturbed orbit always follows an ellipse, which is why this kind of motion is often referred to as a “football orbit”:

\[
x(t) = -\frac{a_e}{2} \cos(\beta) + x_d \\
y(t) = a_e \sin(\beta) + y_d + y_r t \\
z(t) = z_{\text{max}} \sin(\phi) \\
\dot{x}(t) = \frac{a_e}{2} n \sin(\beta) \\
\dot{y}(t) = a_e n \cos(\beta) + y_r \\
\dot{z}(t) = z_{\text{max}} n \cos(\phi)
\]
where:

\[ a_e = 2\sqrt{\left(\frac{x_0}{n}\right)^2 + \left(3x_0 + 2\frac{y_0}{n}\right)^2} \quad (1.16) \]
\[ y_d = y_0 - \frac{\dot{x}_0}{n} \quad (1.17) \]
\[ x_d = 4x_0 + \frac{2y_0}{n} \quad (1.18) \]
\[ y_r = -\frac{3}{2}n\dot{x}_d \quad (1.19) \]
\[ \beta = nt + \tan^{-1}\left(\frac{\dot{x}_0}{3nx_0 + 2y_0}\right) \quad (1.20) \]
\[ z_{\text{max}} = \sqrt{\left(\frac{\dot{z}_0}{n}\right)^2 + z_0^2} \quad (1.21) \]
\[ \phi = nt + \tan^{-1}\left(\frac{\dot{z}_0}{nz_0}\right) \quad (1.22) \]

This parametrization shows that the relative motion of the spacecraft at any time is described by an elliptical path, centered at \((x_d, y_d, 0)\) and drifting at a rate \(y_r\). The projection of this path in the orbital plane is a 2 x 1 ellipse of semimajor axis \(a_e\). The out-of-plane motion, in \(z\), is decoupled from the in-plane motion. The motion in \(z\) is a simple harmonic oscillator.

The secular drift can be set to 0 if \(ny_0 = 2\dot{x}_0\). This leads to non-drifting stationary formations. Four particular cases of stationary formations are generally of interest: the leader-follower formation, the in-plane ellipse formation, the circular formation and the projected circular formation [40].

**Leader-Follower Formation.** If all the formation parameters except \(y_d\) are set to 0, the relative motion between the spacecraft breaks down to \(x(t) = 0\), \(y(t) = y_0\) and \(z(t) = 0\). All the spacecraft are on the same orbital path following each other (Fig. 1.8). This formation is called the leader-follower formation.

**In-plane Ellipse Formation.** If there is no motion in \(z\) and no in-plane drift \((y_r = 0)\), the relative motion is given by a 2 x 1 ellipse in the orbital plane, which is called the in-plane ellipse formation (Fig. 1.9). Both spacecraft evolve in the same orbital
1.5. RELATIVE MOTION THEORIES

plane. The in-track separation between the chief and the center of the deputy relative motion ellipse is given by $y_d$.

**Circular Formation.** If $x^2 + y^2 + z^2 = r^2$, where $r$ is a constant, the deputy has a circular relative motion in the reference frame. If $y_r$ is set to 0, the distance between the chief and the deputy is constant. This formation is called the circular formation.

**Projected Circular Formation.** Finally, if the constraint $y^2 + z^2 = L^2$, where $L$ is a constant, is applied, the projected motion of the deputy in the $y - z$ plane is a circle, which is called the Projected Circular Formation. As seen from Earth, the deputy evolves on a circle (Fig. 1.10). This formation can have several Earth-observation applications.
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Deputy relative motion in y-z plane

Figure 1.10 Projected Circular Formation

The CWH model leads to analytical solutions that provide insight to the relative motion problem and the linearized solution may be used to develop simple control laws. This model is particularly useful for rendezvous maneuvers that have small distances between the spacecraft and short time spans. However, the CWH is only valid if both the chief and the deputy are on circular or near-circular orbits. The effects of eccentricity and perturbations are not taken into account. Therefore, this model cannot be used for long-term orbit propagation or with non-circular orbits.

1.5.2 Unperturbed Elliptical Reference Orbits

The circular reference orbit model yields considerable errors when the eccentricity of the reference orbit grows [25]. Several models have therefore been proposed to model relative motion about elliptical orbits [42, 7].

In order to obtain linear equations of motion, Schaub [42] uses the dynamics of orbit elements rather than Cartesian coordinates. This allows the modelling of large and elliptical formations without any loss of accuracy. Schaub chose to define the dynamics in terms of the classical orbital elements e:

$$
e = \begin{bmatrix} a & e & i & \Omega & \omega & M \end{bmatrix}^T$$ \hspace{1cm} (1.23)

where $a$ is the semimajor axis, $e$ is the orbit eccentricity, $i$ is the orbit inclination, $\Omega$ is the right ascension of the ascending node, $\omega$ is the argument of perigee and $M$ is the mean anomaly. The relative motion is described with the vector of relative orbital elements:

$$\delta e = \begin{bmatrix} \delta a & \delta e & \delta i & \delta\Omega & \delta\omega & \delta M \end{bmatrix}^T$$ \hspace{1cm} (1.24)
1.5. RELATIVE MOTION THEORIES

which is the difference between the orbit element vector of the deputy and the orbit element vector of the chief.

Based on a linear mapping between the relative orbit elements and the deputy coordinates in chief’s LVLH frame, the relative position of the deputy can be obtained:

\[
\begin{align*}
  x(\nu) &= \frac{r}{a} \delta a + \left( \frac{ae}{\eta} \sin \nu \right) \delta M - a \cos \nu \delta e \\
  y(\nu) &= \frac{r}{\eta^3} (1 + e \cos \nu)^2 \delta M + r \delta \omega + \left( \frac{r}{\eta^2} \sin \nu \right) (2 + e \cos \nu) \delta e + r \cos \delta \Omega \\
  z(\nu) &= r (\sin \theta \delta i - \cos \theta \sin \delta \Omega)
\end{align*}
\]

(1.25)

(1.26)

(1.27)

where \( \nu \) is the true anomaly of the chief, \( r \) is the radial distance between the center of the planet and the chief location and \( \eta = \sqrt{1 - e^2} \). In these expressions, absolute orbit elements are those of the chief, while relative orbit elements represent the deputy’s relative states. Similar developments have been done by Lane and Axelrad [27], but with time as independent variable.

Melton [32] also proposed an alternative solution for small-eccentricity orbits. Melton uses a different approach to include the effects of orbit eccentricity. A State-Transition Matrix (STM) is used to provide a closed-form solution to the relative motion problem. The development of the method uses a truncated approximation to order of \( e^2 \). Thus, the method is only valid for \( e \leq 0.3 \). The STM \( \Phi \) maps the relative state vector \( \delta X = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^T \) (relative position and velocity in LVLH frame) at time \( t = 0 \) to the relative state vector at time \( t \).

Melton’s solution is based on the equations of motion of a deputy in a rotating LVLH frame with respect to a reference spacecraft:

\[
\begin{align*}
  \ddot{x} &= \frac{2\mu}{r^3} x + 2\omega(t)\dot{y} + \omega(t)y + \omega^2(t)x + f_x(t) \\
  \ddot{y} &= -\frac{\mu}{r^3} y - 2\omega(t)\dot{x} - \omega(t)x + \omega^2(t)y + f_y(t) \\
  \ddot{z} &= -\frac{\mu}{r^3} z + f_z(t)
\end{align*}
\]

(1.28)

(1.29)

(1.30)

where \( \omega(t) \) in this case is the angular velocity of the rotating frame (not to be confused with the reference orbit argument of perigee) and \( \mu \) is Earth’s gravitational parameter.
This model differs from the CWH model because both the angular velocity \( \omega(t) \) and the radial distance \( r \) of the reference spacecraft are now time-varying as a consequence of the non-zero orbit eccentricity. The dynamics of the system can be written in matrix form:

\[
\delta \dot{X} = A(t)\delta X + f(t)
\]  

(1.31)

where:

\[
A(t) = 
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\frac{2\mu}{r^3} + \omega^2 & \dot{\omega} & 0 & 0 & 2\omega & 0 \\
-\dot{\omega} & -\frac{\mu}{r^3} + \omega^2 & 0 & -2\omega & 0 & 0 \\
0 & 0 & -\frac{\mu}{r^3} & 0 & 0 & 0
\end{bmatrix}
\]

(1.32)

\[
f(t) = 
\begin{bmatrix}
0 \\
0 \\
0 \\
f_x \\
f_y \\
f_z
\end{bmatrix}^T
\]

(1.33)

Therefore, the solution may be expressed as:

\[
\delta X(t) = \Phi(t,0)\delta X(0) = \exp \left[ \int A(t)dt \right] \delta X(0)
\]

(1.34)

However, the closed-form solution cannot be obtained because of the time-varying terms in \( A(t) \). In Ref. [32], Melton uses an expansion in powers of eccentricity of the time-varying terms of \( A(t) \) to expand the matrix \( A(t) \):

\[
A(t) = A_0 + eA_1(t) + e^2A_2(t) + \ldots
\]

(1.35)

A theorem by Poincaré states that the STM can also be expanded:

\[
\Phi(t) = \Phi_0 + e\Phi_1(t) + e^2\Phi_2(t) + \ldots
\]

(1.36)

The elements of \( \Phi_0, \Phi_1 \) and \( \Phi_2 \) are given for both Cartesian and polar coordinate systems in Ref. [32].

Through a convolution integral, this method could also take into account perturbations and control accelerations if an explicit time dependence of \( f \) is known:

\[
\delta X(t) = \Phi(t)\delta X(0) + \int_0^t \Phi(t - \tau)f(\tau)d\tau
\]

(1.37)
This method presents the advantage of providing an approximated closed-form solution for the relative motion for a small-eccentricity orbit. However, the definition of the elements of $\Phi$ is quite lengthy and provides no insight into the dynamics of the system, as opposed to the CWH model. Furthermore, the model is not valid for $e > 0.3$ and cannot cope with perturbations or control accelerations that are not explicit functions of time.

On the other hand, Carter [10] proposed modifications to the well-known Lawden’s equations to remove singularities along the reference orbit. This model predicts motion about unperturbed elliptical reference orbits with the true anomaly as the independent variable:

\[
\begin{align*}
    x(\nu) &= \sin \nu \left( d_1 e + 2d_2 e^2 H(\nu) \right) - \cos \nu \left( \frac{d_2 e}{(1 + e \cos \nu)^2} + d_3 \right) \\
    y(\nu) &= \left[ d_1 + \frac{d_4}{1 + e \cos \nu} + 2d_2 e H(\nu) \right] \\
        &\quad + \sin \nu \left[ \frac{d_3}{1 + e \cos \nu} + d_3 \right] + \cos \nu \left[ d_1 e + 2d_2 e^2 H(\nu) \right] \\
    z(\nu) &= \sin \nu \left[ \frac{d_5}{1 + e \cos \nu} \right] + \cos \nu \left[ \frac{d_6}{1 + e \cos \nu} \right]
\end{align*}
\]

where the constants $d_i$ are defined by initial conditions and the integral $H(\nu)$ is:

\[
H(\nu) = \int_{\nu_0}^{\nu} \frac{\cos \nu}{(1 + e \cos \nu)^3} d\nu
= - (1 - e^2)^{-5/2} \left[ \frac{3eE}{2} - (1 + e^2) \sin E + \frac{e}{2} \sin E \cos E + d_H \right]
\]

where $E$ is the eccentric anomaly and $d_H$ is computed from $H(\nu_0) = 0$.

### 1.5.3 Perturbed Circular Reference Orbits

Some models also take into account orbit perturbations. The most important perturbation encountered for the relative motion problem, and also the most studied, is the perturbation caused by the oblateness of the Earth, referred to as the $J_2$ perturbation. Schweighart and Sedwick [48, 49] modified the classic Clohessy-Wiltshire-Hill model...
to include the orbit-averaged impact of the $J_2$ perturbation on a circular reference orbit. In the development of this model, it is assumed that the dynamics of both the chief and the deputy are under the influence of a spherical gravitational field $g$ and the $J_2$ potential:

$$\dot{r} = g(r) + J_2(r) \quad (1.42)$$

with:

$$g(r) = \frac{\mu}{|r|^3} r \quad (1.43)$$

$$J_2(r) = -\frac{3 J_2 \mu R_e^2}{r^4} \left[ \left( 1 - 3 \sin^2 \theta \sin^2 \varphi \right) x 
+ \left( 2 \sin^2 \theta \sin \varphi \cos \varphi \right) y + \left( 2 \sin \varphi \cos \varphi \sin \varphi \right) z \right] \quad (1.44)$$

where $r$ is the position of the spacecraft in an inertial frame centred on Earth, $\theta$ is the argument of latitude, $\mu$ is the Earth’s gravitational parameter, $R_e$ is Earth equatorial radius and $x$, $y$ and $z$ are the three unit vectors of a rotating LVLH reference frame.

Equation 1.42 can be linearized about a reference position $r_{ref}$:

$$\dot{r} = g(r_{ref}) + \nabla g(r_{ref}) \cdot \rho + J_2(r_{ref}) + \nabla J_2(r_{ref}) \cdot \rho \quad (1.45)$$

where $\rho$ is the position of the spacecraft in a reference LVLH frame centred at $r_{ref}$. In the reference frame rotating with an angular velocity $\omega$ normal to the orbital plane, the dynamics are:

$$\dot{\rho} + 2\omega \times \dot{\rho} + \omega \times (\omega \times \rho) = g(r_{ref}) + \nabla g(r_{ref}) \cdot \rho + J_2(r_{ref}) + \nabla J_2(r_{ref}) \cdot \rho - \ddot{r}_{ref} \quad (1.46)$$

In order to transform equation 1.46 into a constant-coefficient linearized differential equation, the orbit average of $\nabla J_2(r)$ is used:

$$\nabla J_2(r) \approx \frac{1}{2\pi} \int_0^{2\pi} \nabla J_2(r)d\theta = \frac{\mu}{r^3} \begin{bmatrix} 4s & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & -3s \end{bmatrix} \quad (1.47)$$

where:

$$s = \frac{3J_2 R_e^2 [1 + 3 \cos(2\varphi)]}{8r^2} \quad (1.48)$$
In Ref. [48], Schweighart and Sedwick apply this result to the equations of motion of the reference orbit to correct the mean angular velocity, the nodal drift and the cross-track motion of the reference orbit. Their work leads to a modified set of equations of uncontrolled relative motion under the effect of $J_2$ in the LVLH frame:

\begin{align*}
\ddot{x} - 2\left(n\sqrt{1+s}\right)\dot{y} - [5(1+s) - 2]n^2x &= 0 \\
\ddot{y} + 2\left(n\sqrt{1+s}\right)\dot{x} &= 0 \\
\ddot{z} + q^2z - 2\eta q (q + \phi) &= 0
\end{align*}

where $q$, $l$ and $\phi$ are constants defined by the out-of-plane initial conditions and $n = \sqrt{\mu/a^3}$ is the unperturbed orbit angular velocity. Assuming $\dot{x}_0$ and $\dot{y}_0$ have been chosen to remove any residual drift and offset terms, a closed-form solution is obtained:

\begin{align*}
x &= x_0 \cos (nt\sqrt{1-s}) + \frac{\sqrt{1-s}}{2\sqrt{1+s}}y_0 \sin (nt\sqrt{1-s}) \\
y &= -\frac{2\sqrt{1+s}}{2\sqrt{1-s}}x_0 \sin (nt\sqrt{1-s}) + y_0 \cos (nt\sqrt{1-s}) \\
z &= (lt + m) \sin (qt + \phi)
\end{align*}

with:

\begin{align*}
\dot{x}_0 &= \frac{ny_0}{2} \frac{1-s}{\sqrt{1+s}} \\
\dot{y}_0 &= -2nx_0 \sqrt{1+s}
\end{align*}

The linearized $J_2$ model has shown to be quite accurate and easy to implement. The main modelling error comes from the use of the average value of the gradient of $J_2$. However, the effects of eccentricity are not included in the model.

### 1.5.4 Perturbed Elliptical Reference Orbits

The most challenging problem is to consider an elliptical and perturbed reference orbit. The most accurate way to model this problem is of course with numerical models. In this case, solutions to the relative motion problem are obtained through numerical integration of the non-linear dynamics equations [26, 4, 12]. The most complete model
for the motion of a spacecraft including the effect of the $J_2$ harmonic may be defined in Cartesian coordinates $x^I$, $y^I$ and $z^I$ in the inertial frame, as described in Ref. [4]:

\[ \ddot{x}^I = -\frac{\mu x^I}{r^3} \left[ 1 - \frac{3J_2}{2} \left( \frac{R_e}{r} \right)^2 \left( \frac{5 (z^I)^2}{r^2} - 1 \right) \right] + f_x \]  
\[ \ddot{y}^I = \frac{y^I}{x^I} \ddot{x}^I + f_y \]  
\[ \ddot{z}^I = -\frac{\mu z^I}{r^3} \left[ 1 + \frac{3J_2}{2} \left( \frac{R_e}{r} \right)^3 \left( 3 - \frac{5 (z^I)^2}{r^2} \right) \right] + f_z \]

where $R_e = 6.3781 \times 10^6$ m is Earth's equatorial radius, $\mu = 3.986 \times 10^{14}$ m^3/s^2 is Earth's gravitational constant, $J_2 = 1.08264 \times 10^{-3}$ is the $J_2$ harmonic constant and $f_x$, $f_y$ and $f_z$ are control accelerations or accelerations caused by perturbations, such as solar radiation pressure, atmospheric drag or other gravity field harmonics. An analytical solution cannot be obtained to this problem. The relative motion is obtained by integrating through time the trajectory of all the spacecraft and by differentiating their positions.

In Ref. [37], the equation of the relative position $\rho$ of spacecraft $j$ relative to spacecraft $i$ in generalized coordinates (a generalized form of Eq. 1.46) is developed:

\[ \dot{\rho}_{ij} + 2\omega \times \rho_{ij} + \omega \times (\omega \times \rho_{ij}) + \dot{\omega} \times \rho_{ij} = \left( \frac{Q_j}{m_j} - \frac{Q_i}{m_i} \right) + \left( \frac{U_{q_j}}{m_j} - \frac{U_{q_i}}{m_i} \right) \]  

where $Q_j$ is the generalized force vector on the spacecraft $j$ and $U_{q_j}$ the gravitational force acting on the spacecraft $j$ orbiting around a reference orbit rotating with an angular velocity $\omega$. This model expresses in a compact form the equations of relative motion of a formation composed of a large number of spacecraft. Nevertheless, the prediction of relative motion necessitates numerical integration.

Numerical methods are not well suited for autonomous on-board applications because they typically require a lot of computing effort and provide no insight into efficient guidance and control solutions. Gim and Alfriend [13] solve the problem by proposing a state transition matrix that provides a time explicit solution to the relative motion problem on a $J_2$-perturbed elliptical orbit. Their STM $\Phi_{J_2}(t, t_0)$ maps the relative position state vector 

\[ \delta X = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^T \]  

at $t_0$ to the relative position state vector.
1.5. RELATIVE MOTION THEORIES

at a given time $t$:

$$\delta X(t) = \Phi_{J_2}(t, t_0)\delta X(t_0)$$  \hspace{1cm} (1.61)

The transformation between the relative orbital elements vector $\delta e$ (the difference between the spacecraft orbit element vector and the reference orbit element vector) and the relative state vector may be written as:

$$\delta X(t) = [A(t) + \alpha B(t)] \delta e$$  \hspace{1cm} (1.62)

where $\alpha$ is a constant and $B(t)$ contains the terms perturbed by $J_2$. If $\Phi_e$ is the STM for the orbit elements, such as $\delta e(t) = \Phi_e(t, t_0)\delta e(t_0)$, $\Phi_{J_2}$ may be re-written through algebraic manipulations as:

$$\Phi_{J_2}(t, t_0) = [A(t) + \alpha B(t)] \Phi_e(t, t_0) [A(t_0) + \alpha B(t_0)]^{-1}$$  \hspace{1cm} (1.63)

To build the STM $\Phi_e$, the authors use propagation of the mean orbit elements. As stated earlier, mean orbit elements are elements from which short-term oscillations caused by the $J_2$ perturbation have been removed. They only show secular drift, and this drift rate is constant. The matrix $\Phi_e(t, t_0)$ is therefore defined as a function of the transformation matrix $D(t)$ between the relative mean elements $\delta \bar{e}$ and the relative osculating elements $\delta e$ and the STM $\tilde{\Phi}_e(t, t_0)$ for the mean elements only:

$$\delta e(t) = D(t)\delta \bar{e}$$

$$= D(t)\tilde{\Phi}_e(t, t_0)\delta \bar{e}(t_0)$$

$$= D(t)\tilde{\Phi}_e(t, t_0)D^{-1}(t_0)\delta e(t_0)$$  \hspace{1cm} (1.64)

$$\Phi_e(t, t_0) = D(t)\tilde{\Phi}_e(t, t_0)D^{-1}(t_0)$$  \hspace{1cm} (1.65)

Thus, $\Phi_{J_2}(t, t_0)$ becomes:

$$\Phi_{J_2}(t, t_0) = \Sigma(t)D(t)\tilde{\Phi}_e(t, t_0)D^{-1}(t_0)\Sigma^{-1}(t_0)$$  \hspace{1cm} (1.66)

where:

$$\Sigma(t) = A(t) + \alpha B(t)$$  \hspace{1cm} (1.67)

The elements of $\Sigma(t)$, $\Sigma^{-1}(t)$, $\tilde{\Phi}_e(t, t_0)$ and $D(t)$ are given in Ref. [13].
This model provides an accurate solution to the problem, assuming linearization conditions are respected. However, even though the model is fully analytical, the elements of the state transition matrices remain quite complex, the states of the reference trajectory still need to be numerically computed and matrix products and inversions remain.

Another variation of this solution was suggested by Schaub [42]. This solution linearizes the dynamics in the relative mean orbit element space only to predict the impact of $J_2$ perturbations in terms of relative mean coordinates. Once the dynamics of the relative mean orbit elements are known, this result can be substituted back into the relative orbit elements linearized mapping (Eq. 1.25 to 1.27). As shown in Ref. [46], assuming only $J_2$ perturbations are present, the rates of the mean orbit elements are constant:

\[
\frac{d\alpha}{dt} = 0 \quad \text{(1.68)} \\
\frac{d\delta}{dt} = 0 \quad \text{(1.69)} \\
\frac{d\delta}{dt} = 0 \quad \text{(1.70)} \\
\frac{d\Omega}{dt} = -\frac{3}{2} J_2 n \left( \frac{R_e}{p} \right)^2 \cos i \quad \text{(1.71)} \\
\frac{d\omega}{dt} = \frac{3}{4} J_2 n \left( \frac{R_e}{p} \right)^2 (5 \cos^2 i - 1) \quad \text{(1.72)} \\
\frac{dM_0}{dt} = \frac{3}{4} J_2 n \left( \frac{R_e}{p} \right)^2 \sqrt{1 - e^2} (3 \cos^2 i - 1) \quad \text{(1.73)}
\]

If these element drift rates are linearized with respect to relative orbit elements, mean relative orbit element drift can be estimated for any given true anomaly. The $J_2$ perturbation will cause a drift of the relative ascending node, relative argument of perigee and relative mean anomaly:

\[
\delta\Omega(\nu) = \delta\Omega(\nu_0) + \epsilon \delta \kappa_\Omega [M(\nu) - M_0] \quad \text{(1.74)} \\
\delta\omega(\nu) = \delta\omega(\nu_0) + \epsilon \delta \kappa_\omega [M(\nu) - M_0] \quad \text{(1.75)} \\
\delta M_0(\nu) = \delta M_0(\nu_0) + \epsilon \delta \kappa_{M_0} [M(\nu) - M_0] \quad \text{(1.76)}
\]
where:

\[ \delta \kappa_\Omega = \frac{7}{4} \cos i \left( \frac{\delta a}{a} \right) - \frac{2e}{\eta^2} \cos i \delta e + \frac{1}{2} \sin i \delta i \]  
(1.77)

\[ \delta \kappa_\omega = -\frac{7}{8} \left( 5 \cos^2 i - 1 \right) \left( \frac{\delta a}{a} \right) + \frac{e}{\eta^2} \left( 5 \cos^2 i - 1 \right) \delta e - \frac{5}{4} \sin (2i) \delta i \]  
(1.78)

\[ \delta \kappa_M = -\frac{7}{8} \eta \left( 3 \cos^2 i - 1 \right) \left( \frac{\delta a}{a} \right) + \frac{3e}{4\eta} \left( 3 \cos^2 i - 1 \right) \delta e - \frac{3\eta}{4} \sin (2i) \delta i \]  
(1.79)

\[ \epsilon = 3J_2 \left[ \frac{r}{a(1-e^2)} \right]^2 \]  
(1.80)

The mean anomaly difference can be updated with the new \( \delta M_0 \):

\[ \delta M(\nu) = \delta M_0 - \frac{3}{2} (M(\nu) - M_0) \frac{\delta a}{a} \]  
(1.81)

This model is in closed-form and fully analytical. However, it only predicts relative motion in the mean orbit element space. The mapping between osculating and mean orbit elements (and between mean and osculating orbit elements) still needs to be performed on the orbit elements vector if the "true" (osculating) elements are needed. Ref. [46] provides a first-order mapping between mean and osculating orbital elements that showed to be accurate for the propagation of relative motion.

### 1.5.5 Gauss Variational Equations

The Gauss Variational Equations (GVEs) are used to model the impact of applied forces on orbit elements. This impact is given as a function of acceleration components in the orbital frame \( \{O_r, O_t, O_h\} \), where \( O_r \) is aligned with the position vector \( r \) of the spacecraft, \( O_h \) is in the direction of the orbital momentum (normal to orbital plane) and \( O_t \) completes the right-hand frame (Fig. 1.11). The reader shall note that the orbital frame axes \( O_r, O_t \) and \( O_h \) are identical to the \( x, y \) and \( z \) axes of a LVLH frame centered on this spacecraft. In the orbital frame, the radial acceleration \( a_r \), the transverse acceleration \( a_t \) and the normal acceleration \( a_h \) have the following impact on the semimajor axis \( a \), the eccentricity \( e \), the inclination \( i \), the right ascension of the ascending node \( \Omega \), the argument of perigee \( \omega \) and the mean anomaly \( M \) [46]:
\[
\frac{da}{dt} = \frac{2a^2}{h} \left(e \sin \nu a_r + \frac{p}{r} a_t\right) \tag{1.82}
\]

\[
\frac{de}{dt} = \frac{1}{h} \left\{p \sin \nu a_r + [(p + r) \cos \nu + re] a_t\right\} \tag{1.83}
\]

\[
\frac{d\dot{\nu}}{dt} = \frac{r \cos \theta}{h} a_h \tag{1.84}
\]

\[
\frac{d\dot{\Omega}}{dt} = \frac{r \sin \theta}{h \sin i} a_h \tag{1.85}
\]

\[
\frac{d\omega}{dt} = \frac{1}{he} \left[-p \cos \nu a_r + (p + r) \sin \nu a_t\right] - \frac{r \sin \theta \cos i}{h \sin i} a_h \tag{1.86}
\]

\[
\frac{dM}{dt} = n + \frac{b}{ah e} \left[(p \cos \nu - 2re) a_r - (p + r) \sin \nu a_t\right] \tag{1.87}
\]

where \(p = a(1 - e^2)\) is the orbit semi-latus rectum, \(r = a/(1 + e \cos \nu)\) the current orbital radius, \(h = \sqrt{\mu/m}\) the orbit angular momentum and:

\[
b = r_p \sqrt{\frac{1 + e}{1 - e}} \tag{1.88}
\]

where \(r_p = a(1 - e)\) is the radius at perigee. In the case of a short-duration maneuver, as in the case of an impulsive thrust for example, the impact of the three components \(\Delta V_r\), \(\Delta V_t\) and \(\Delta V_h\) of a velocity impulse on the orbit elements can be easily obtained
assuming an infinitesimal duration of the impulse:

\[
\Delta a = \frac{2a^2}{h} \left( e \sin \nu \Delta V_r + \frac{p}{r} \Delta V_i \right) \quad (1.89)
\]

\[
\Delta e = \frac{1}{h} \left\{ p \sin \nu \Delta V_r + [(p + r) \cos \nu + re] \Delta V_i \right\} \quad (1.90)
\]

\[
\Delta i = \frac{r \cos \theta}{h} \Delta V_h \quad (1.91)
\]

\[
\Delta \Omega = \frac{r \sin \theta}{h \sin i} \Delta V_h \quad (1.92)
\]

\[
\Delta \omega = \frac{1}{he} \left\{ -p \cos \nu \Delta V_r + (p + r) \sin \nu \Delta V_i \right\} - \frac{r \sin \theta \cos i}{h \sin i} \Delta V_h \quad (1.93)
\]

\[
\Delta M = \frac{b}{ahe} \left\{ (p \cos \nu - 2re) \Delta V_r - (p + r) \sin \nu \Delta V_i \right\} \quad (1.94)
\]

These equations have been developed for absolute orbit elements, as opposed to relative orbit elements. Nevertheless, they can be applied to relative orbit elements, as a change in absolute orbit elements relates to exactly the same change in relative orbit elements if the reference is uncontrolled. The GVEs are therefore the model of choice to predict the impact of a control acceleration or impulse on the relative motion of a deputy.

### 1.5.6 Summary

A simple closed-form analytical model of relative motion that includes the \(J_2\) perturbation for elliptical reference orbits is yet to be developed. The two analytical models of relative motion for perturbed elliptical orbits (the Gim-Alfriend STM and the relative mean orbit element propagation model) are not yet fully adapted for on-board implementation. Both methods require the propagation of the reference trajectory forward in time to perform the osculating to mean and mean to osculating orbit elements mapping. In both cases, the mean to osculating and the osculating to mean mapping is to be performed separately on every spacecraft of the formation. Furthermore, the Gim-Alfriend STM necessitates some numerical matrix inversions. A fully analytical model in a STM form would have the advantage of being readily applicable to all the elements of the formation.
CHAPTER 1. INTRODUCTION

1.6 Relative Orbit Control Methods

Once the maneuver has been defined, the control system needs to take the spacecraft from its initial position to its targeted location while minimizing the propellant consumption, the duration of the maneuver and/or the risk of collision. This section presents the different algorithms that have been suggested to track the reference states provided by the guidance system. These algorithms include traditional linear controllers, non-linear continuous controllers, impulsive feedback controllers and control algorithms based on on-line numerical optimization.

1.6.1 Linear Optimal Control

By using the CWH linearized model of relative motion, traditional linear control can be applied to formation flying. The main advantages of linear control is that it is a well-known method, with measurable performance and robustness assuming the linearization conditions are valid.

For example, a Linear-Quadratic Regulator (LQR) can be tuned to compute the control acceleration vector $u$ to compensate for a state vector error $\Delta x$ through a feedback gain matrix $K$ such as:

$$ u = -K\Delta x $$

where $K$ is chosen to minimize a cost function $J$:

$$ J = \int_0^\infty \left[ (\Delta x)^T Q (\Delta x) + u^T R u \right] dt $$

where $Q$ and $R$ are positive definite matrices. Ref. [52] evaluates the performance and robustness of a LQR to maintain a planar formation on a circular orbit. Preliminary simulation results by the candidate have also shown that the LQR can be applied to in-plane and out-of-plane maneuvers with reasonable fuel consumption even with elliptical orbits. However, as is the case with many other systems, increasing the controller gains (decreasing the control weight $R$), will reduce the response time of the
controller but with an increased fuel cost. This controller seems promising for long-term formation keeping, which only implies small maneuvers.

An optimal reconfiguration maneuver of two spacecraft assuming CWH dynamics is developed in Ref. [38] also using optimal control. The main conclusion of the work is that a balanced fuel-optimal maneuver of two spacecraft on unperturbed circular orbits is achieved through equal and opposite acceleration of both spacecraft. However, these conclusions do not necessarily apply to elliptical and perturbed orbits. In fact, in Ref. [25], it is demonstrated that assuming a circular orbit, even when $e = 0.005$, leads to significant increase of fuel cost because the spacecraft “fights” the natural dynamics to keep the same relative trajectory as it would in a circular orbit.

### 1.6.2 Continuous Mean Orbit Elements Feedback Control Laws

The continuous mean orbit elements feedback control law, as described in Refs. [47] and [46], controls the current mean orbit element vector of the spacecraft toward the desired mean orbit element vector. By defining the error in terms of orbit elements, it is possible to “cooperate” with the physics of orbital dynamics. Acting directly on the orbit elements allows the control of specific orbit elements at specific moments of the orbit to increase the fuel efficiency of the algorithm. For example, it is much more fuel efficient to correct an inclination error at equator than at the pole, while an error in the ascending node is easier to compensate near the poles. By carefully choosing the gain matrix of the controller, these effects can be accounted for.

Let $e_{osc} = \begin{bmatrix} a/R_e & e & i & \Omega & \omega & M \end{bmatrix}^T$ be the vector of osculating orbit elements. A semimajor axis normalized with the equatorial radius is used to facilitate the choice of controller gains. Using GVEs (Section 1.5.5), the time-derivative of $e_{osc}$ can be obtained straightforwardly:

\[
\dot{e}_{osc} = \begin{bmatrix} 0 & 0 & 0 & 0 & n \end{bmatrix}^T + B(e_{osc})u
\]  

(1.97)
where \( n = \sqrt{\mu/a^3} \) is the mean motion of the orbit, \( u = \begin{bmatrix} u_r & u_t & u_h \end{bmatrix}^T \) is the control vector and the control influence matrix \( B \) is:

\[
B(e_{osc}) = \begin{bmatrix}
\frac{2a^2e \sin \nu}{hR_e} & \frac{2a^2p}{hrR_e} & 0 \\
\frac{p \sin \nu}{h} & \frac{(p + r) \cos \nu + re}{h} & 0 \\
0 & 0 & \frac{r \cos \theta}{h} \\
0 & 0 & \frac{r \sin \theta}{h \sin i} \\
-\frac{p \cos \nu}{he} & \frac{(p + r) \sin \nu}{he} & -\frac{r \sin \theta \cos i}{h \sin i} \\
\eta(p \cos \nu - 2re) & \eta(p + r) \sin \nu & 0
\end{bmatrix}
\] (1.98)

Let \( e = \begin{bmatrix} a/R_e & \bar{e} & \bar{i} & \bar{\omega} & \bar{M} \end{bmatrix}^T \) be the vector of mean orbit elements and \( \xi \) the transformation between osculating and mean elements:

\[
e = \xi e_{osc}
\] (1.99)

Using a first-order approximation:

\[
\dot{e} = A(e) + \frac{\partial \xi}{\partial e_{osc}} B(e_{osc}) u
\] (1.100)

with \( A \) being the time-derivative of mean orbit elements under the influence of \( J_2 \) (Eq. 1.68 to 1.73):

\[
A = \begin{bmatrix}
0 \\
0 \\
0 \\
-\frac{3}{2}J_2 \left( \frac{R_e}{p} \right)^2 n \cos i \\
\frac{3}{4}J_2 \left( \frac{R_e}{p} \right)^2 n \left( 5 \cos^2 i - 1 \right) \\
n + \frac{3}{4}J_2 \left( \frac{R_e}{p} \right)^2 \eta n \left( 3 \cos^2 i - 1 \right)
\end{bmatrix}
\] (1.101)

In Ref. [46], the study of the transformation function between mean and osculating elements leads to the conclusion that \( \partial \xi/\partial e_{osc} \) is practically an identity matrix, with
diagonal terms of order of $J_2$ or smaller. Therefore, the dynamics of the mean orbit elements can be approximated as:

$$
\dot{e}_{osc} = A(e) + B(e_{osc})u
$$

(1.102)

Even though the Gauss variational equations provide the effect of actuators on osculating orbit elements, it is assumed that a change of an osculating element will imply the same change of the corresponding mean orbit element. It is also to be noted that the numerical difference between $B(e_{osc})$ and $B(e)$ is negligible, so that:

$$
\dot{e}_{osc} = A(e) + B(e)u
$$

(1.103)

Given the actual set of orbit elements $e$ and the desired set of orbit elements $e_d$, the control law seeks to minimize the orbit element error $\Delta e$:

$$
\Delta e = e - e_d
$$

(1.104)

If the desired relative orbit elements are $J_2$-invariant (no control is required to maintain the relative orbit), then:

$$
\dot{e}_d = A(e_d)
$$

(1.105)

and the dynamics of the error can be written as:

$$
\Delta \dot{e} = A(e) + B(e)u - A(e_d)
$$

(1.106)

Several control laws can be derived from equation 1.106. In Ref. [46], it is suggested to use Lyapunov control theory to develop a feedback law. Let $V$ be a positive definite Lyapunov function of the tracking error $\Delta e$:

$$
V = \frac{1}{2} (\Delta e)^T \Delta e
$$

(1.107)

Taking the derivative of $V$, one finds:

$$
\dot{V} = \Delta e^T \Delta \dot{e} = \Delta e^T [A(e) + B(e)u - A(e_d)]
$$

(1.108)

Forcing $\dot{V}$ to be negative definite yields:

$$
\dot{V} = -\Delta e^T P \Delta e
$$

(1.109)
where $P$ is a positive definite $6 \times 6$ matrix. The two previous equations lead to the following constraint for the stability of the closed-loop system:

$$B(e)u = - [A(e) - A(e_d)] - P\Delta e$$  \hspace{1cm} (1.110)

Equation 1.110 is over-determined. There are six elements to control, but only three components in the control vector. Several strategies can be used to resolve this issue and define a feedback control law. The first one is to use a least-square type inverse (or Pseudo-inverse) to solve for $u$:

$$u = - (B(e)^T B(e))^{-1} B(e)^T [A(e) - A(e_d) + P\Delta e]$$  \hspace{1cm} (1.111)

However, the preceding control law is not guaranteed to satisfy the stability criterion of equation 1.110 because of the nature of the pseudo-inverse. If $P$ is large enough, another solution could be to drop the $A(e) - A(e_d)$ term in equation 1.110, which leads to:

$$u = - [B(e)^T B(e)]^{-1} B(e) P\Delta e$$  \hspace{1cm} (1.112)

In this case, the asymptotic stability of the controller can be demonstrated (see Ref. [46]). In both previous control laws, the gain matrix $P$ can be carefully chosen to "cooperate" with orbital dynamics. It does not have to be a constant. It only has to be positive definite. As suggested in Ref. [46], one solution is to define $P$ as a diagonal matrix with time-varying terms:

$$P = I \begin{bmatrix}
P_{e_0} + P_{e_1} \cos N \psi \\
P_{h_0} + P_{h_1} \cos N \nu \\
P_{\nu_0} + P_{\nu_1} \cos N \Omega \\
P_{\Omega_0} + P_{\Omega_1} \sin N \theta \\
P_{w_0} + P_{w_1} \sin N \phi \\
P_{M_0} + P_{M_1} \sin N \nu
\end{bmatrix}$$  \hspace{1cm} (1.113)

where $N$ is an even integer. This causes gains on particular orbit elements error to be high when the latter are the most controllable and to become negligible when the corresponding orbit element is not controllable.
Another simplified solution is to define a $3 \times 3$ positive definite feedback matrix $K$ such as:

$$u = -KB(e)^T\Delta e$$  \hspace{1cm} (1.114)

In this case, $B(e)$ acts as the time-varying gain matrix. When the controllability of a given orbit element is high, the corresponding entry in $B(e)$ will be high. Once again, the stability of this control law can be demonstrated assuming $A(e) - A(e_d)$ is negligible (Ref. [46]).

### 1.6.3 Continuous Cartesian Coordinates Feedback Control Laws

If the desired trajectory is described as an inertial position $r_d = [x^I \ y^I \ z^I]^T$ and an inertial velocity $\dot{r}_d$, a control feedback law based on Cartesian coordinates errors can be used, as described in Ref. [46].

The dynamics of the spacecraft can be modelled as:

$$\dot{r} = f(r) + u$$  \hspace{1cm} (1.115)

where $u$ is the control acceleration vector and $f(r)$ the uncontrolled dynamics, including perturbations caused by $J_2$:

$$f(r) = -\frac{\mu}{r^3} \left[ r - J_2\frac{3}{2} \left( \frac{R_e}{r} \right)^2 \left( \begin{array}{c} 5x^I \left( \frac{z^I}{r} \right)^2 - x^I \\ \frac{y^I}{r} \right) \\ 5y^I \left( \frac{z^I}{r} \right)^2 - y^I \\ 5z^I \left( \frac{z^I}{r} \right)^2 - 3z^I \end{array} \right]$$  \hspace{1cm} (1.116)

Let $\Delta r = r - r_d$ be the position error vector and $\Delta \dot{r} = \dot{r} - \dot{r}_d$ be the velocity error. Let $V$ be a Lyapunov function such as:

$$V = \frac{1}{2} \Delta \dot{r}^T \Delta \dot{r} + \frac{1}{2} \Delta r^T K_1 \Delta r$$  \hspace{1cm} (1.117)

where $K_1$ is a positive definite matrix. Taking the derivative of $V$ leads to:

$$\dot{V} = \Delta \dot{r}^T (\Delta \ddot{r} - \Delta \dot{r}_d + K_1 \Delta r)$$  \hspace{1cm} (1.118)
If the desired orbit can be maintained without control, $\dot{V}$ can be re-written as:

$$
\dot{V} = \Delta \hat{r}^T [f(r) - f(r_d) + u + K_1 \Delta r]
$$  \hspace{1cm} (1.119)

Setting $V$ to be negative definite:

$$
\dot{V} = -\Delta \hat{r}^T K_2 \Delta \dot{r}
$$  \hspace{1cm} (1.120)

where $K_2$ is a $3 \times 3$ positive definite matrix leads to the following control law:

$$
u = -[f(r) - f(r_d)] - K_1 \Delta r - K_2 \Delta \dot{r}
$$  \hspace{1cm} (1.121)

The asymptotic stability of this control law is demonstrated in Ref. [46]. A similar control law, but adaptive to slowly varying spacecraft masses is presented in Ref. [11].

### 1.6.4 Hybrid Feedback Control Law

The hybrid feedback law [46] uses desired states defined as a set of orbit element differences with a reference orbit, while the tracking errors are Cartesian relative coordinates errors. The main advantage of that method is that the controller uses inputs that are easily measured (relative position and velocity in LVLH frame) while the reference is defined as orbit elements, which is more conveniently expressed than rapidly evolving Cartesian coordinates.

In Hill coordinates ($X = \begin{bmatrix} x & y & z \end{bmatrix}^T$ and $V = \dot{X}$), the linearized equations for relative elliptical orbits are [32]:

$$
\dot{X} = V
$$  \hspace{1cm} (1.122)

$$
\dot{V} = A_1 X + A_2 V + u
$$  \hspace{1cm} (1.123)
where:

\[
A_1 = \begin{bmatrix}
\frac{2\mu}{r^3} + \dot{\theta}^2 & \ddot{\theta} & 0 \\
-\ddot{\theta} & \ddot{\theta}^2 - \frac{\mu}{r^3} & 0 \\
0 & 0 & -\frac{\mu}{r^3}
\end{bmatrix}
\]  
(1.124)

\[
A_2 = \begin{bmatrix}
0 & 2\dot{\theta} & 0 \\
-2\dot{\theta} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]  
(1.125)

and \( \theta \) is the argument of latitude. The argument of latitude acceleration is given by:

\[
\ddot{\theta} = -2\frac{\mu}{r^3} (q_1 \sin \theta - q_2 \cos \theta)
\]  
(1.127)

where \( q_1 = e \cos \omega \) and \( q_2 = e \sin \omega \).

The relative orbit tracking errors \( \Delta X \) and \( \Delta V \) are computed with respect to a desired position \( X_d \) and velocity \( V_d \) in the LVLH frame:

\[
\Delta X = X - X_d
\]  
(1.128)

\[
\Delta V = V - V_d
\]  
(1.129)

where the desired states are obtained through a linear mapping between Hill coordinates and the set of orbit elements \( e = [a \theta i q_1 q_2 \Omega]^T \):

\[
\begin{bmatrix}
X_d \\
V_d
\end{bmatrix} = A(e)\delta e_d
\]  
(1.130)

where \( \delta e_d \) is the vector of orbit element differences between the desired orbit and the orbit elements corresponding to the origin of the LVLH frame. Let the control law be:

\[
u = \dot{V}_d - A_1X - A_2V - K\Delta X - P\Delta V
\]  
(1.131)

If the reference trajectory can be maintained without any control:

\[
\dot{V}_d = A_1X_d + A_2V_d
\]  
(1.132)
then the control law can be written as:

\[ u = - [A_1 + K, A_2 + P] \left( \begin{bmatrix} X \\ V \end{bmatrix} - A(e) \delta e_d \right) \]  

(1.133)

It is demonstrated in Ref. [46] that this control law is asymptotically stabilizing.

### 1.6.5 Impulsive Feedback Control Law

The previous control methods use continuous thrust to maintain the formation. The Impulsive Feedback Controller (IFC) uses instead thrust impulses at specific moments of the orbit to maintain or reconfigure a formation [46, 43]. This controller was designed in order to perform any small orbit element correction

\[ \Delta e = \begin{bmatrix} \Delta a \\ \Delta e \\ \Delta i \\ \Delta \Omega \\ \Delta \omega \\ \Delta M \end{bmatrix}^T \]

within one orbit with only three impulses. More specifically, it was suggested as a way to perform corrections on one orbit element while minimizing the impact on the other orbit elements. Given the initial set of orbit elements:

\[ e = \begin{bmatrix} a \\ e \\ i \\ \Omega \\ \omega \\ M \end{bmatrix}^T \]

the IFC thus proposes a way to reach the desired set of orbit elements \( e_d = e + \Delta e \) with three impulses.

From looking at GVEs (Eq. 1.89 and Eq. 1.94), it is clear that both inclination error and ascending node error can be corrected with only one out-of-plane impulse. The required out-of-plane velocity impulse \( \Delta V_h \) for inclination and ascending node corrections is:

\[ \Delta i = \frac{r \cos \theta}{h} \Delta V_h \]

(1.134)

\[ \Delta \Omega = \frac{r \sin \theta}{h \sin i} \Delta V_h \]

(1.135)

If both \( \Delta i \) and \( \Delta \Omega \) are to be performed, it can be shown that the optimal way to perform both corrections is through a single impulse at the critical latitude angle \( \theta_c \) given by:

\[ \theta_c = \arctan \left( \frac{\Delta \Omega \sin i}{\Delta i} \right) \]  

(1.136)
The corresponding velocity impulse thus becomes:

\[ \Delta V_h = \frac{h}{r} \sqrt{\Delta i^2 + \Delta \Omega^2 \sin i^2} \]  

(1.137)

The velocity impulse of equation 1.137 only impacts \( i, \Omega \) and \( \omega \). Substituting equation 1.137 in equation 1.93, the corresponding change \( \Delta \omega_{V_h} \) in \( \omega \) is given by:

\[ \Delta \omega_{V_h} = -\cos i \Delta \Omega \]  

(1.138)

This change in \( \omega \) will be accounted for in the computation of the corresponding velocity impulse.

The four other orbit elements changes (\( \Delta a, \Delta e, \Delta \omega \) and \( \Delta M \)) can be performed with only two in-plane burns at perigee and at apogee.

The argument of perigee and the mean anomaly are corrected as a pair with two radial impulses, one at perigee (\( \Delta V_{rp} \)) and one at apogee (\( \Delta V_{ra} \)). Based on GVEs and including the impact of the ascending node correction (Eq. 1.138), the implied changes in \( \omega \) and \( M \) by the two impulses are the following:

\[ \Delta \omega = -\frac{p}{he} (\Delta V_{rp} - \Delta V_{ra}) - \Delta \Omega \cos i \]  

(1.139)

\[ \Delta M = \frac{\eta}{he} [(p - 2r_p e) \Delta V_{rp} - (p + 2r_a e) \Delta V_{ra}] \]  

(1.140)

with \( \eta = \sqrt{1 - e^2} \) and where \( r_p \) is the radius at perigee, \( r_a \) the radius at apogee and \( p \) the semi-latus rectum. Solving for \( \Delta V_{rp} \) and \( \Delta V_{ra} \) leads to:

\[ \Delta V_{rp} = -\frac{na}{4} \left[ \frac{(1 + e)^2}{\eta} \left( \Delta \omega + \Delta \Omega \cos i \right) + \Delta M \right] \]  

(1.141)

\[ \Delta V_{ra} = -\frac{na}{4} \left[ \frac{(1 - e)^2}{\eta} \left( \Delta \omega + \Delta \Omega \cos i \right) + \Delta M \right] \]  

(1.142)

The last two elements, \( a \) and \( e \) are corrected through two tangential burns, one at perigee (\( \Delta V_{tp} \)) and one at apogee (\( \Delta V_{ta} \)). The impact of those burns on \( a \) and \( e \) is respectively:

\[ \Delta a = \frac{2a}{h} \left( \frac{p}{r_p} \Delta V_{tp} + \frac{p}{r_a} \Delta V_{ta} \right) \]  

(1.143)

\[ \Delta e = \frac{1}{h} \left[ (p + r_p + r_p e) \Delta V_{tp} + (-p - r_a + r_a e) \Delta V_{ta} \right] \]  

(1.144)
The preceding equations assume that $\Delta a$ and $\Delta e$ are small, so that $a$ and $e$ can be assumed to be constant between the two burns. Solving for $\Delta V_{tp}$ and $\Delta V_{ta}$ yields:

\[ \Delta V_{tp} = \frac{nan}{4} \left( \frac{\Delta a}{a} + \frac{\Delta e}{1 + e} \right) \tag{1.145} \]
\[ \Delta V_{ta} = \frac{nan}{4} \left( \frac{\Delta a}{a} - \frac{\Delta e}{1 - e} \right) \tag{1.146} \]

To implement this algorithm, the orbit element errors have to be computed at an arbitrary point on the orbit and kept constant as long as the required velocity impulses have not all been performed. Obviously, this will cause an inaccuracy in the magnitude of the impulses. This algorithm is thus to be used iteratively for a certain number of orbits before all six orbit elements are properly corrected.

If only one or two orbit elements have to be corrected, this algorithm provides essentially optimal results. However, if all six orbit elements have to be corrected, this algorithm provides a near-optimal solution with a fuel cost of only a few percent over the optimal multi-impulse solution [46]. This method can be used for formation reconfiguration. However, this method would not be suitable for formation maintenance where formation accuracy is required because the response time is in the order of one orbit period.

In Ref. [1], an impulsive feedback control law is developed for an orbit of small eccentricity, but allowing only tangential and out-of-plane thrust impulses. In Ref. [30], the optimal impulsive maneuver for in-plane unperturbed circular formation is computed. In Ref. [29], the effect of $J_2$ is also taken into account for in-plane reconfiguration of formations orbiting on a circular orbit.

### 1.6.6 Numerical Methods

Numerical methods have also been proposed for formation flying control. Numerical methods require on-line numerical optimization, typically to evaluate the future states of the spacecraft, as opposed to the previously presented analytical methods, that use analytical solutions to provide control commands.
In Ref. [6], a model predictive controller is proposed. This controller uses GVEs and mean orbit elements to model the $J_2$ perturbed dynamics. In Ref. [34], Mishne proposes an impulsive velocity corrections algorithm, but takes into account the effect of drag and oblateness of the Earth. The solution procedure requires on-line numerical integration for propagation of the states.

Finally, the use of highly-powerful numerical optimization algorithms, such as genetic algorithms [2] has also been studied. This type of algorithm computes the optimal velocity impulses required for a given maneuver. However, the high computational load of this kind of method and the risk of not converging to a feasible solution preclude an on-board implementation.

**1.6.7 Summary**

Linear, non-linear, analytical, numerical, impulsive and continuous methods have been applied to formation flying control. The choice of the control method is highly dependent of the mission design. Traditional linear continuous controllers are well-suited for close formations on circular orbits. Non-linear control laws, based on Cartesian coordinates or orbit elements can cope with high-eccentricity reference orbits. These are continuous feedback laws that require thrusters that can fire in a continuous and variable fashion. Impulsive feedback control laws provide more fuel-optimal responses. However, the corrections can take several orbits before they are completed. Numerical methods allow more flexibility. They can be time-optimized, fuel-optimized and can provide multi-impulse firing schemes. However, on-board implementation of those methods is precluded by the (most of the time) heavy required computational load.

For on-board applications, analytical methods are obviously better suited. However, none of the analytical methods presented here can guarantee the optimality of the maneuver and a reasonable response time. Moreover, the performance of these controllers relies heavily on a careful tuning of the controller gains, which is a very cumbersome task.
1.7 Project Objectives

The research project is thus oriented toward the development of autonomous guidance and control algorithms for formation flying spacecraft. From the literature review presented earlier, three main challenges conspicuously remain in order to obtain a fully autonomous guidance and control loop. These three challenges are:

1. The development of a simple analytical model of relative motion for perturbed elliptical orbits. Several models have been developed for circular or near-circular orbits. However, an accurate and simple model of relative motion that encompasses the effects of the $J_2$ perturbation for highly elliptical reference orbits is yet to be developed.

2. The development of an autonomous guidance algorithm that performs real-time trade-offs between conflicting requirements. Formation flying spacecraft mainly face four different types of requirements: formation accuracy, individual fuel consumption, balancing of the fuel consumption among the spacecraft and collision avoidance. Obviously all these requirements are contradictory. Typically, the trade-offs between these requirements is performed on the ground by the mission operators. However, in an autonomous scenario, the trade-off has to be performed by the spacecraft based on its situational awareness.

3. The development of an optimal or near-optimal analytical feedback control algorithm. The relative motion control algorithms previously available in the literature and suitable for on-board autonomy cannot guarantee the fuel-optimality of the maneuvers and rely on an onerous gain fine-tuning process. In a context where the total amount of fuel on-board is the main driver of the total formation operational life-time, it becomes obvious that a control algorithm that can perform the same maneuvers with a smaller amount of fuel is desired.

The main objective of this research project is therefore to identify and implement solutions for these three aspects of formation flying guidance and control, solutions which
are to remain suitable for on-board applications and minimize the dependency on ground support. Ultimately, this should lead to a fully autonomous guidance and control loop.

1.8 Thesis Outline

The objectives of the project are achieved firstly by developing tools that can be used by an autonomous guidance system: an analytical model of relative motion about perturbed elliptical orbits (Chapter 2) and the Fuel-Equivalent Space theory (Chapter 3). Chapter 2 describes a new analytical model of relative motion for elliptical reference orbits while taking into account the secular drift caused by the $J_2$ perturbation. Chapter 3 then describes a mathematical tool, the Fuel-Equivalent Space, in which similar displacements on any axis lead to a similar fuel cost. This approach greatly simplifies the fuel minimization problem by mapping it into a simpler geometric distance minimization problem.

In turn, a neighbouring optimal feedback control law is developed to perform the maneuver planned by the guidance algorithms in the most fuel-efficient way (Chapter 4). This feedback controller is in a semi-analytic form and guarantees near-optimal maneuvering for any spacecraft in the vicinity of the reference trajectory for which the controller is synthesized.

Finally, Chapter 5 shows that these three developments can be tied in together to form a completely autonomous guidance and control loop. Through the relative value of only three scalar gains, the user can perform trade-offs between formation accuracy, fuel minimization and balancing of the fuel spending among the members of the formation. The guidance and control system consequently autonomously selects the best location for each spacecraft in the formation, plans the maneuver for every spacecraft and executes it in the most fuel-efficient way.
PART I

Autonomous Formation Flying

Guidance Tools
CHAPTER 2

Linearized Dynamics of Formation Flying Spacecraft on a $J_2$-Perturbed Elliptical Orbit


Abstract

A linearized set of equations of relative motion about a $J_2$-perturbed elliptical reference orbit is developed. This model uses analytical relations that are well suited for on-board applications. The inclusion of the $J_2$ perturbation in a simple analytical model can lead to formation flying guidance and control algorithms that make use of the natural $J_2$-induced relative motion to perform maneuvers instead of constantly compensating for this perturbation. The model uses the linearized differential drift rate of mean orbit elements to predict the impact of the $J_2$ perturbation on relative osculating spacecraft motion. It analytically provides the relative motion in Hill coordinates at any given true anomaly using only the initial osculating relative orbit elements and the initial orbit elements of the reference trajectory. A linear time-varying state-space form of the model is also presented. Simulation results show that relative motion prediction remains accurate over several orbits.
2.1 Introduction

Formation flying of spacecraft has been identified a key technology for the 21st century. There is a trend toward replacing large expensive spacecraft by a group of smaller and cheaper spacecraft. Two of the main advantages of formation flying are reconfigurability and an increased robustness to system failures. The main drawback of formation flying spacecraft is an increased complexity of the complete system of spacecraft. This is particularly true for the guidance, navigation and control system, whose complexity grows rapidly with the number of spacecraft in the formation.

There is however, at the same time, an increasing need for autonomy to decrease the cost of ground support. Ground support operations are a non-negligible part of the cost of a mission, especially for small and low-cost scientific exploration missions. Therefore, the guidance and control system needs to perform autonomous decisions and trade-offs in real-time to decrease the tasks that need to be performed by the ground segment and make formation flying affordable. Moreover, to increase the robustness to single spacecraft failure, the guidance and control of the formation needs to be decentralized. This is especially challenging when the number of spacecraft in the formation becomes large.

Such guidance and control systems require accurate but simple models of reality in their algorithms. Models have to be accurate enough to prevent unnecessary fuel expenditure and simple enough to allow on-board implementation. If perturbation models are included in the on-board model of reality, natural motion induced by the perturbations can be used to support maneuvers. If these perturbations are not included, the guidance and control system will most likely compensate for these perturbations, therefore leading to an unnecessary fuel expenditure.

The most widely used relative orbital motion model is the Clohessy-Wiltshire-Hill model [54, 40]. This model provides a time-explicit closed-form analytical solution to relative motion problem for circular unperturbed orbits. Lovell and Tragesser [30] reparametrized this model and demonstrated that the in-plane and out-of-plane, non-
drifting, relative motion about a circular unperturbed orbit always follows an ellipse centered on the reference orbit, hence the name "football orbit".

However, assuming a circular reference orbit yields considerable errors when the eccentricity of the reference orbit grows [25]. Several models have therefore been proposed to model relative motion about unperturbed elliptical orbits [42, 7, 25, 58]. In a recent publication, Lane and Axelrad [27] develop a time-explicit closed-form solution and study the relative motion for bounded relative elliptical orbits. Melton [32] also proposed an alternative solution for small-eccentricity orbits.

Some models also take into account orbit perturbations. The most important perturbation encountered for the relative motion problem, and also the most studied, is the perturbation caused by the oblateness of the Earth, referred to as the \( J_2 \) perturbation. Schweighart and Sedwick [48, 49] modified the classic Clohessy-Wiltshire-Hill model to include the orbit-averaged impact of the \( J_2 \) perturbation on a circular reference orbit.

The most challenging problem is to consider an elliptical and perturbed reference orbit. The most accurate way to model this problem is of course with numerical models [26, 4]. In this case, solutions to the relative motion problem are obtained through numerical integration of the dynamics equations. However, numerical methods are not well suited for autonomous on-board applications because they typically require a lot of computing effort. Few publications actually provide an analytical solution to the relative motion around elliptical reference orbits taking into account the \( J_2 \) perturbation.

Gim and Alfriend [13] solve the problem by proposing a state transition matrix that provides a time explicit solution for the relative motion about a \( J_2 \)-perturbed elliptical orbit. This model provides an accurate solution to the problem. However, even though the model is fully analytical, the elements of the state transition matrices remain quite complex, the states of the reference trajectory still need to be numerically computed and matrix products and inversions remain. On the other hand, Schaub studies the relative motion about elliptical reference orbits under \( J_2 \) perturbation with very simple expressions using classical orbit elements [42]. However, this analysis is only performed in
the mean orbit element space. This model cannot be written readily into a state transition matrix form as the mapping between instantaneous, or osculating, orbit elements remains to be done.

The purpose of this paper is therefore to develop an analytical state transition matrix that accurately models relative motion about elliptical reference orbits under $J_2$ perturbation, while using simpler expressions and without the need to numerically propagate the states of the reference trajectory. It builds upon the approach of Schaub [42], but bridges the gap between osculating relative motion and relative mean orbit element drift. Desired formation relative dynamics will be described in terms of osculating, or "actual", relative dynamics, which is why it is relevant to describe the relative motion in terms of osculating elements instead of mean elements. This simplified model is oriented toward an on-board implementation for mission scenarios where computational power is limited, such as low-cost scientific missions.

The proposed model uses a geometric approach, similar to the work of Gim-Alfriend [13] but with certain simplifying assumptions. The model neglects variations in the short-periodic relative motion induced by the $J_2$ perturbation between the deputy and the chief, but includes a osculating to mean orbit elements mapping. In other words, it "adds" the relative mean orbit element drift to the natural osculating elements Keplarian dynamics, neglecting the impact of short-period variations on the relative motion. This simplification is made at the cost of a prediction error as large as the short-periodic terms variations between the deputy and the chief. For two spacecraft orbiting very close from one another, this error will remain small as the short-period oscillations caused by the $J_2$ perturbation will be the same for both spacecraft. However, in all cases, this error will remain bounded even for long-term prediction. The main advantage of this approach is that the states of the reference trajectory at the true anomaly where the relative dynamics need to be known are not required. All the elements of the state transition matrix are computed from the initial position of the reference trajectory and the true anomaly for which the relative motion needs to be predicted. Models that take into account short-period variations [13] will need the states of the reference at the
2.2. MAPPING BETWEEN HILL FRAME AND ORBIT ELEMENTS

final time to do an accurate mapping between the mean elements and the osculating elements at this location.

The model presented here makes use of the classical orbit elements, singular if the reference orbit is circular. The main reason is when the $J_2$ perturbation is modeled, the use of classical elements radically simplifies the expressions as only 3 of the 6 orbit elements experience secular drift. Obviously, the main drawback of the classical elements is that the model cannot be used for circular reference orbits. However, for missions with large eccentricity orbits, such as ESA's currently planned Proba-3 mission, this model can be used without any fear of going through a singularity.

This paper is organized as follows. Section 2.2 presents a linear mapping between orbit elements and coordinates in the curvilinear Hill frame. Section 2.3 presents how the relative drift between the chief and the deputy can be modelled through the use of mean orbit elements. Section 2.4 shows how the flight time can be estimated for a given true anomaly if the $J_2$ perturbation is considered. Then, section 2.5 combines those three results to yield a completely linearized set of equations describing the relative motion of spacecraft on a $J_2$-perturbed elliptical orbit. Section 2.6 translates the model into a linear time-varying state-space model convenient for the design of control laws. Finally, section 2.7 presents simulation results and compares the accuracy of the model with the exact non-linear model and the elliptical unperturbed model.

2.2 Linearized Mapping Between Hill Frame Coordinates and Orbit Elements

The linear mapping between the relative orbit elements and the coordinates in a local-vertical local-horizontal Hill frame is realized by differentiating the position of a spacecraft with a given set of orbit elements in an inertial frame and then by rotating this result in the Hill frame. This mapping has been already presented by Schaub [46, 42], and is reproduced here using an alternative development, to serve as the starting point.
for the extension proposed here. A similar result, but with non-singular elements has also been obtained by Gim and Alfriend [13].

Let \( \mathbf{e} = \begin{bmatrix} a & e & i & \Omega & \omega & M \end{bmatrix}^T \) be the orbit element vector of a spacecraft, where \( a \) is the orbit semimajor axis, \( e \) the orbit eccentricity, \( i \) the orbit inclination, \( \Omega \) the right ascension of the ascending node, \( \omega \) the argument of perigee and \( M \) the mean anomaly. The Cartesian coordinates \( \mathbf{r}^I \) of the spacecraft in an Earth-centered inertial frame \( \mathcal{I} \) can easily be obtained from \( \mathbf{e} \):

\[
\mathbf{r}^I = r \begin{bmatrix} 
(\cos \Omega \cos \omega - \sin \Omega \sin \omega \cos i) \cos \nu - (\cos \Omega \sin \omega + \sin \Omega \cos \omega \cos i) \sin \nu \\
(\sin \Omega \cos \omega + \cos \Omega \sin \omega \cos i) \cos \nu + (- \sin \Omega \sin \omega - \cos \Omega \cos \omega \cos i) \sin \nu \\
\sin \omega \sin i \cos \nu + \cos \omega \sin i \sin \nu 
\end{bmatrix}
\]

(2.1)

where \( \nu \) is the true anomaly and \( r \) is the orbit radius:

\[
r = \frac{a(1-e^2)}{1+e \cos \nu}
\]

(2.2)

Even though \( \nu \) is not explicitly part of \( \mathbf{e} \), it can be obtained iteratively from the mean anomaly and the orbit eccentricity through the well-known relations:

\[
M = E - e \sin E \\
\tan \frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}
\]

(2.3) \hspace{1cm} (2.4)

The impact on the inertial position of a small difference in orbit elements \( \delta \mathbf{e} = [\delta a, \delta e, \delta i, \delta \Omega, \delta \omega, \delta M]^T \) can be estimated with a first-order approximation:

\[
\Delta \mathbf{r}^I = \frac{\partial \mathbf{r}^I}{\partial r} \delta r + \frac{\partial \mathbf{r}^I}{\partial \Omega} \delta \Omega + \frac{\partial \mathbf{r}^I}{\partial \omega} \delta \omega + \frac{\partial \mathbf{r}^I}{\partial i} \delta i + \frac{\partial \mathbf{r}^I}{\partial \nu} \delta \nu
\]

(2.5)

using [46]:

\[
\delta r = \frac{r}{a} \delta a + \frac{ae \sin \nu}{\eta} \delta M - a \cos \nu \delta e
\]

(2.6)

\[
\delta \nu = \frac{(1 + e \cos \nu)^2}{\eta^2} \delta M + \frac{\sin \nu}{\eta^2} (2 + e \cos e) \delta e
\]

(2.7)

where \( \eta = \sqrt{1-e^2} \) leads to \( \Delta \mathbf{r}^I \) fully expressed as a function of \( e \) and \( \delta e \). The position increment can finally be expressed in a common Local-Vertical Local-Horizontal...
(LVLH) Hill frame $\mathcal{H} = \{\hat{x}, \hat{y}, \hat{z}\}$, where $\hat{x}$ is a unit vector pointing in the direction of the spacecraft position $r$ (Earth-centered position), $\hat{z}$ is a unit vector normal to the orbital plane pointing in the direction of the orbit angular momentum, and $\hat{y}$ completes the right-hand frame. To do so, the relative position of the spacecraft in the inertial frame is multiplied by the rotation matrix $C_{\mathcal{HT}}$ from $\mathcal{I}$ to $\mathcal{H}$:

$$\Delta r^\mathcal{H} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = C_{\mathcal{HT}} \Delta r^\mathcal{I}$$

(2.8)

where:

$$C_{\mathcal{HT}} = \begin{bmatrix} \cos(\nu) \cos(\Omega) - \sin(\nu) \sin(\Omega) \cos(i) & \cos(\nu) \sin(\Omega) + \sin(\nu) \cos(\Omega) \cos(i) & \sin(\nu) \sin(i) \\ -\sin(\nu) \cos(\Omega) - \cos(\nu) \sin(\Omega) \cos(i) & -\sin(\nu) \sin(\Omega) + \cos(\nu) \cos(\Omega) \cos(i) & \cos(\nu) \sin(i) \\ \sin(\Omega) \sin(i) & -\cos(\Omega) \sin(i) & \cos(i) \end{bmatrix}$$

(2.9)

The final result is therefore the position of a deputy spacecraft in the Hill frame centered on a chief spacecraft as a function of the chief orbit elements $e$ and the orbit element difference $\delta e$ between the deputy and the chief:

$$x(\nu) = \frac{r}{a} \delta a - a \cos \nu \delta e + \frac{ae \sin \nu}{\eta} \delta M$$

(2.10)

$$y(\nu) = \frac{r \sin \nu (2 + e \cos \nu)}{\eta^2} \delta e + r \cos i \delta \Omega + r \delta \omega + \frac{r (1 + e \cos \nu)^2}{\eta^3} \delta M$$

(2.11)

$$z(\nu) = r \sin(\nu + \omega) \delta i - r \sin i \cos(\nu + \omega) \delta \Omega$$

(2.12)

### 2.3 Orbit Element Drift on a $J_2$-Perturbed Elliptical Orbit

If both orbits are Keplerian, setting $\delta a = 0$ ensures both spacecraft have the same orbital period. This leads to a constant and non-drifting $\delta e$. The prediction of relative motion can therefore be realized by sweeping $\nu$ from 0 to $2\pi$ in Eq. 2.10 to 2.12 to get the
relative motion for a complete orbit. However, if perturbations are encountered, $\delta e$ will not remain constant. The orbit element differences will evolve with $\nu$. If only the $J_2$ perturbation is considered, orbit elements will experience short-period oscillations and secular drift.

A common way to predict the effect of the $J_2$ perturbation on spacecraft motion is to use mean orbit element propagation. Mean orbit elements are orbit elements from which short-period oscillations have been removed. They only show secular drift which can be easily expressed analytically.

Let $\bar{e} = \begin{bmatrix} \bar{a} & \bar{e} & \bar{i} & \bar{\Omega} & \bar{\omega} & \bar{M} \end{bmatrix}^T$ be the vector of mean orbit elements of the chief. It has been shown [8] that only $\bar{\Omega}, \bar{\omega}$ and $\bar{M}$ will have non-zero secular drift rate caused by $J_2$:

\[
\begin{align*}
\dot{\bar{a}} &= 0 & (2.13) \\
\dot{\bar{e}} &= 0 & (2.14) \\
\dot{\bar{i}} &= 0 & (2.15) \\
\dot{\bar{\Omega}} &= -\frac{3}{2} J_2 \bar{\eta} \left( \frac{R_e}{\bar{p}} \right)^2 \cos \bar{i} & (2.16) \\
\dot{\bar{\omega}} &= \frac{3}{4} J_2 \bar{\eta} \left( \frac{R_e}{\bar{p}} \right)^2 \left( 5 \cos^2 \bar{i} - 1 \right) & (2.17) \\
\dot{\bar{M}} &= \bar{\eta} + \frac{3}{4} J_2 \bar{\eta} \left( \frac{R_e}{\bar{p}} \right)^2 \bar{\eta} \left( 3 \cos^2 \bar{i} - 1 \right) & (2.18)
\end{align*}
\]

where $\bar{\eta}$ and $\bar{\eta}$ are respectively the mean motion and $\eta$ computed with mean eccentricity, $R_e$ is Earth’s equatorial radius and $\bar{p} = a \bar{\eta}^2$ is the semilatus rectum based on mean orbit elements. A first-order mapping between actual elements (commonly referred to as “osculating”) and mean elements is provided by Schaub [46].

It is the difference in drift rates that is most relevant for formation flying as it has a long-term impact on the spacecraft relative motion. The impact of $\delta e$ on those drift
2.3. ORBIT ELEMENT DRIFT ON A $J_2$-PERTURBED ELLIPTICAL ORBIT

Rates differences can be approximated by differentiating the previous equations:

$$
\delta \dot{\Omega} = \frac{\partial \dot{\Omega}}{\partial a} \delta a + \frac{\partial \dot{\Omega}}{\partial e} \delta e + \frac{\partial \dot{\Omega}}{\partial i} \delta i
$$

$$
\delta \dot{\omega} = \frac{\partial \dot{\omega}}{\partial a} \delta a + \frac{\partial \dot{\omega}}{\partial e} \delta e + \frac{\partial \dot{\omega}}{\partial i} \delta i
$$

$$
\delta \dot{M} = \frac{\partial \dot{M}}{\partial a} \delta a + \frac{\partial \dot{M}}{\partial e} \delta e + \frac{\partial \dot{M}}{\partial i} \delta i
$$

with the partial derivatives given by:

$$
\frac{\partial \dot{\Omega}}{\partial a} = \frac{21}{a} C \cos i
$$

$$
\frac{\partial \dot{\Omega}}{\partial e} = \frac{24e}{\eta^2} C \cos i
$$

$$
\frac{\partial \dot{\Omega}}{\partial i} = 6C \sin i
$$

$$
\frac{\partial \dot{\omega}}{\partial a} = -\frac{21}{2a} C \left(5 \cos^2 i - 1 \right)
$$

$$
\frac{\partial \dot{\omega}}{\partial e} = \frac{12e}{\eta^2} C \left(5 \cos^2 i - 1 \right)
$$

$$
\frac{\partial \dot{\omega}}{\partial i} = -15C \sin(2i)
$$

$$
\frac{\partial \dot{M}}{\partial a} = \frac{-3\dot{\eta}}{2a} - \frac{\eta}{4a} C \left(63 \cos (2i) - 21 \right)
$$

$$
\frac{\partial \dot{M}}{\partial e} = \frac{9\dot{\eta}}{\eta} C \left(3 \cos^2 i - 1 \right)
$$

$$
\frac{\partial \dot{M}}{\partial i} = -9\eta C \sin(2i)
$$

where:

$$
C = \frac{J_2 \eta R_\varepsilon^2}{4p^2}
$$

If the impact of relative short-period oscillations is neglected, it can be assumed that the evolution of the osculating orbit element differences will only be caused by the relative secular drift of mean orbit elements. Hence, the mean orbit element drift rate difference is a way to estimate the orbit element differences of the drifting elements $\Omega$, $\omega$, and $M$. 

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\( \omega \) and \( M \):

\[
\begin{align*}
\delta a(\tau) &= \delta a_0 \\ 
\delta e(\tau) &= \delta e_0 \\ 
\delta i(\tau) &= \delta i_0 \\ 
\delta \Omega(\tau) &= \delta \Omega_0 + \delta \dot{\Omega} \tau \\ 
\delta \omega(\tau) &= \delta \omega_0 + \delta \dot{\omega} \tau \\ 
\delta M(\tau) &= \delta M_0 + \delta \dot{M} \tau
\end{align*}
\]  

(2.32)  

(2.33)  

(2.34)  

(2.35)  

(2.36)  

(2.37)

where \( \delta a_0, \delta e_0, \delta i_0, \delta \Omega_0, \delta \omega_0 \) and \( \delta M_0 \) are the initial osculating orbit element differences (at time \( t_0 \)) and \( \tau \) is the elapsed flight time since \( t_0 \).

2.4 Estimation of the Flight Time

The evolution of the relative orbit elements is known as a function of the elapsed flight time. However, the proposed model needs to use only \( \nu \) as independent variable. Therefore, the flight time \( \tau \) needs to be estimated as a function of the true anomaly \( \nu \).

The flight time can be estimated assuming that the evolution of mean anomaly with time is known. In an unperturbed environment, the mean anomaly rate is constant and equal to the mean motion \( n \). However, in a \( J_2 \)-perturbed environment, the mean anomaly rate is not equal to the mean motion. Neglecting the effect of short-period and long-period oscillations (thus only assuming secular drift) leads to Eq. 2.18. This assumption will leave short-period errors in the estimation of flight time, but will not cause any long term drifting error for a \( J_2 \)-perturbed orbit.

The relationship between eccentric anomaly \( E \) and mean anomaly \( M \) is the well-known Kepler equation:

\[
M = E - e \sin E
\]

(2.38)
where the eccentric anomaly can be expressed as a function of $\nu$:

$$E = 2 \arctan \left[ \sqrt{\frac{1-e}{1+e}} \tan \left( \frac{\nu}{2} \right) \right]$$ (2.39)

Equations 2.38 and 2.39 can be used straightforwardly to get the estimated mean anomaly $M(\nu)$ for a given true anomaly $\nu$. The elapsed time $\tau$ is obtained assuming a constant mean anomaly rate since $t_0$:

$$\tau = \frac{2\pi N_{\text{orb}} + M(\nu) - M_0}{\dot{M}}$$ (2.40)

where $N_{\text{orb}}$ is the number of orbits that the spacecraft has performed, if the model is to be used to perform long-term prediction over several orbits. This estimated flight time, combined with the estimated relative drift rate of orbit elements, allows the estimation of relative osculating elements as a function of the true anomaly.

The relevance of considering the $J_2$ perturbation in the evaluation of flight time can easily be demonstrated. Figure 2.1 shows the flight time estimation error if $a_0 = R_e + 1000\text{km}$, $e_0 = 0.1$, $i_0 = \pi/4$, and $\Omega_0 = \omega_0 = M_0 = 0$ for a spacecraft evolving in a $J_2$-perturbed environment. The flight time estimation error is shown for 10 orbits. No secular error can be observed when considering the secular drift caused by $J_2$ on the mean anomaly rate. However if one assumes $\dot{M} = n$, the error reaches 100 s (nearly 1.5% of the orbital period) after 10 orbits and keeps growing. The short-period oscillations found in both cases are due to the neglected short-period $J_2$ perturbations.

### 2.5 Linearized Equations of Motion

The results of the previous sections can be combined to predict drifting spacecraft relative motion. Substituting $\delta a, \delta e, \delta i, \delta \Omega, \delta \omega$ and $\delta M$ for $\delta a(\tau), \delta e(\tau), \delta i(\tau), \delta \Omega(\tau), \delta \omega(\tau)$ and $\delta M(\tau)$ and substituting $e$ for $e_0 + \dot{e}_0 \tau$ into Eq. 2.10 to 2.12 leads to a set of equations...
CHAPTER 2. DYNAMICS OF FORMATION FLYING SPACECRAFT

Estimated Flight Time Error

Figure 2.1 Estimated Flight Time Error

that takes into account the differential drift caused by \( J_2 \):

\[
\begin{align*}
    x(\nu) &= \frac{a_0}{a_0} \delta a_0 + \frac{a_0 e_0 \sin \nu}{\eta_0} \left( \delta M_0 + \dot{\delta} M \right) - a_0 \cos \nu \delta e_0 \\
    y(\nu) &= \frac{r(\nu)}{\eta_0^2} \delta e_0 + r(\nu) \cos \nu \left( \delta \Omega_0 + \dot{\delta} \Omega \right) + r(\nu) \left( \delta \omega_0 + \dot{\delta} \omega \right) \\
    &\quad + \frac{r(\nu) (1 + e_0 \cos \nu)^2}{\eta_0^3} \left( \delta M_0 + \dot{\delta} M \right) \\
    z(\nu) &= r(\nu) \sin(\nu + \omega_0 + \dot{\omega} \tau) \delta i_0 - r(\nu) \sin \nu \cos(\nu + \omega_0 + \dot{\omega} \tau) \left( \delta \Omega_0 + \dot{\delta} \Omega \right)
\end{align*}
\]

(2.41) (2.42) (2.43)

where \( e_0 = \left[ a_0 \ e_0 \ i_0 \ \Omega_0 \ \omega_0 \ M_0 \right]^T \) are the osculating orbit elements of the reference orbit and \( \dot{e}_0 = \left[ 0 \ 0 \ \dot{\Omega} \ \dot{\omega} \ \dot{M} \right]^T \) are the corresponding mean element drift rates.

Even though Eq. 2.41 to 2.43 may appear to be simplistic statements at first, they in fact represent the crux of the advantage of the main assumption upon which this model is built. The \( \delta \Omega_0, \delta \omega_0 \) and \( \delta M_0 \) terms are osculating relative orbit elements, while \( \delta \dot{\Omega}, \delta \dot{\omega} \) and \( \delta \dot{M} \) are mean relative orbit element drift rates. This assumes that relative motion between osculating elements only shows secular drift and no short-period oscillations. This will of course lead to an inevitable bounded prediction error in the model, but all the terms of Eq. 2.41 to 2.43 can be expressed only from osculating relative orbit
elements and initial osculating orbit elements of the reference trajectory, thus avoiding the need to numerically propagate the states of the chief.

For this to be realized, $\delta \hat{\Omega}$, $\delta \hat{\omega}$ and $\delta \hat{M}$, currently function of $\delta a$, $\delta e$ and $\delta i$, need to be expressed as a linear function of the osculating element differences. This can be done by studying the mapping function $\bar{e} = \xi e$ between the mean and osculating elements. The matrix $\partial \xi / \partial e$ is approximately an identity matrix with off-diagonal of order of $J_2$ or smaller [46]. It is thus reasonable to assume that a small increment of an osculating orbit element will cause the same change of the corresponding mean element. However it has been noted that this assumption is not valid for $a$, for which the off-diagonal terms are non-negligible. Therefore, $\delta a$ has to be approximated with a linearization of the function $\bar{a} = \xi_a e$. Schaub [46] provides a first-order mapping between osculating and mean elements that can be used to approximate $\delta a$:

$$
\bar{a} = \xi_a = a - a J_2 \left( \frac{R_a}{a} \right)^2 \left[ (3 \cos^2 i - 1) \left( \frac{a}{r} \right)^3 - \frac{1}{\eta^3} \right] + 3 \left( 1 - \cos^2 i \right) \left( \frac{a}{r} \right)^3 \cos (2\omega + 2\nu)
$$

This function can be linearized about the orbit $e$:

$$
\delta \bar{a} = \frac{\partial \xi_a}{\partial a} \delta a + \frac{\partial \xi_a}{\partial e} \delta e + \frac{\partial \xi_a}{\partial i} \delta i + \frac{\partial \xi_a}{\partial \omega} \delta \omega + \frac{\partial \xi_a}{\partial \nu} \delta \nu \tag{2.45}
$$

with $\delta \nu$ given by Eq. 2.7. The analytical expressions of the partial derivatives in Eq. 2.45 are developed in Appendix 2.9.

The relative velocity can be obtained straightforwardly by differentiating Eq. 2.41 to 2.43 with respect to time and taking $\dot{t} = 1$. For that purpose, only $r, \nu, \Omega, \omega, M$ and $\tau$
are considered to be functions of time:

\[
\dot{x}(\nu) = \frac{r}{a_0} \delta a_0 + \frac{a_0 e_0 \nu \cos \nu}{\eta} (\delta M_0 + \delta \dot{M} \tau) + \frac{a_0 e_0 \sin \nu}{\eta} \delta \dot{M} + a_0 \nu \sin \nu \delta e_0 \tag{2.46}
\]

\[
\ddot{y}(\nu) = \frac{1}{\eta^2} \left[ \dot{r} \sin \nu (2 + e_0 \cos \nu) + r \dot{\nu} (2 + e_0 \cos \nu) \cos \nu - r e_0 \nu \sin^2 \nu \right] + r \cos i_0 \left( \delta \Omega_0 + \delta \dot{\Omega} \tau \right) + r \cos \omega_0 \delta \omega + \dot{r} \left( 1 + e_0 \cos \nu \right)^2 \left( \delta M_0 + \delta \dot{M} \tau \right) - 2r e_0 \nu \sin \nu \left( 1 + e_0 \cos \nu \right) \left( \delta M_0 + \delta \dot{M} \tau \right) + r \left( 1 + e_0 \cos \nu \right)^2 \delta \dot{M} \tag{2.47}
\]

\[
\dot{z}(\nu) = \dot{r} \sin (\nu + \omega_0 + \dot{\omega} \tau) \delta i_0 - r \cos (\nu + \omega_0 + \dot{\omega} \tau) (\dot{\nu} + \dot{\omega}) \delta i_0 - r \cos (\nu + \omega_0 + \dot{\omega} \tau) \sin i_0 \left( \delta \Omega_0 + \delta \dot{\Omega} \tau \right) + r \sin (\nu + \omega_0 + \dot{\omega} \tau) (\dot{\nu} + \dot{\omega}) \sin i_0 \left( \delta \Omega_0 + \delta \dot{\Omega} \tau \right) - r \cos (\nu + \omega_0 + \dot{\omega} \tau) \sin i_0 \delta \dot{\Omega} \tag{2.48}
\]

where:

\[
\dot{r}(\nu) = \frac{a_0 e_0 \sin \nu}{\eta} \dot{M} \tag{2.49}
\]

\[
\dot{\nu}(\nu) = \frac{(1 + e_0 \cos \nu)^2}{\eta^3} \dot{M} \tag{2.50}
\]

Collecting Eq. 2.41 to 2.48 and substituting \( \delta \dot{\Omega}, \delta \dot{\omega}, \delta \dot{M}, \delta a, \delta e \) and \( \delta i \) by their equivalent function of the initial reference orbit \( e_0 \) and initial offset \( \delta e_0 \) leads to a very convenient way of expressing the model:

\[
\delta X(\nu) = \Phi(e_0, \nu) \delta e_0 \tag{2.51}
\]

where \( \delta X = \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} & x & y & z \end{bmatrix}^T \). The elements of \( \Phi \) are given in Appendix 2.10. Given an initial reference orbit \( e_0 \), the relative dynamics of a spacecraft with a small orbit element offset \( \delta e_0 \) can be predicted through analytical equations for any point of the orbit \( \nu \) considering a \( J_2 \)-perturbed orbit.

In summary, the steps required to predict the relative motion are:

1. Define the initial reference orbit state vector \( e_0 \) and the the true anomaly \( \nu \) for which the formation is needed.
2.5. LINEARIZED EQUATIONS OF MOTION

2. Translate $e_0$ into the mean orbit element space [46]: $\bar{e}_0 = \xi e_0$.

3. Compute the mean reference orbit element drift with Eq. 2.13 to 2.18 from $\bar{e}_0$.

4. Compute the mean orbit element drift rate partial derivatives at $\bar{e}_0$ (Eq. 2.22 to 2.30)

5. Compute the $\xi_a$ partial derivatives at $e_0$ with the results of Appendix 2.9.

6. Estimate the flight time $\tau$ at $\nu$ from Eq. 2.38 to 2.40.

7. Compute the elements of the matrix $\Phi(e_0, \nu)$ found in Appendix 2.10.

The relative Hill coordinates $\delta X(\nu)$ for any initial relative position $\delta e_0$ can then be obtained through $\delta X = \Phi(e_0, \nu) \delta e_0$. Assuming $\Phi$ is non-singular, the model can also be numerically inverted to provide the initial required orbit element differences given a desired configuration at a specific point $\nu$ of the orbit:

$$\delta e_0 = [\Phi(e_0, \nu)]^{-1} \delta X(\nu) \quad (2.52)$$

This result is useful in the sense that it provides the current $\delta e_0$ required to reach without any further control effort a desired formation $\delta X$ at a point $\nu$, considering all spacecraft are under the influence of the $J_2$ perturbation. The difference between the result of Eq. 2.52 and the actual orbit element differences represent the maneuver that is to be performed to achieve the formation $\delta X$ at the desired true anomaly but following natural motion.

The matrix $\Phi$ will become singular as eccentricity $e$ tends toward 0 because a $\omega$ or a $M$ offset cannot be differentiated for a perfectly circular orbit. Furthermore, when $e$ is close to 0, $\omega$ can move very quickly around the orbit, which means that large $\delta \omega$, far beyond the validity limit of the linearized model, can be encountered. The model also fails if $i = 0$ because $\Omega$ cannot be defined.

For circular orbits, the $J_2$ linearized model of Schweighart and Sedwick [48, 49] is better suited to predict relative motion. For the zero inclination case, an unperturbed model
would prove to be sufficiently accurate because equatorial orbits are weakly affected by the $J_2$ perturbation, which can approximately be modelled by an equivalent increased gravity in this case. This model would need to use a different set of elements, such as the equinoctial elements, because $\Omega$ cannot be defined for an equatorial orbit.

### 2.6 State-space Model

The results of Section 2.5 can also be expressed in time-varying linear state-space dynamic model of the form:

$$\delta \dot{e} = A(e_c) \delta e + B(e_c) u$$

(2.53)

where $e_c = \begin{bmatrix} a_c & e_c & i_c & \Omega_c & \omega_c & M_c \end{bmatrix}^T$ is the orbit element vector of the reference, or the chief (that can be any element of the formation or simply a virtual point in space), and $\delta e$ the orbit element offset with respect to the reference. The control vector $u$ is composed of the radial control acceleration $u_r$, the tangential control acceleration $u_\theta$, and out-of-plane control acceleration $u_h$, such that:

$$u = \begin{bmatrix} u_r \\ u_\theta \\ u_h \end{bmatrix}$$

(2.54)

Assuming the chief follows a $J_2$-perturbed uncontrolled motion, the matrix $A$ depicts the relative drift of the orbit element caused by the natural $J_2$ perturbation while the $B$ matrix links the deputy control accelerations to its relative orbit element dynamics.
Since only $\Omega$, $\omega$ and $M$ will experience relative drift, the matrix $A$ is filled by expanding Eq. 2.19 to 2.21:

$$A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial \Omega}{\partial a} & \frac{\partial \Omega}{\partial e} & \frac{\partial \Omega}{\partial \Omega} & 0 & 0 & 0 \\
\frac{\partial \omega}{\partial a} & \frac{\partial \omega}{\partial e} & \frac{\partial \omega}{\partial \omega} & 0 & 0 & 0 \\
\frac{\partial M}{\partial a} & \frac{\partial M}{\partial e} & \frac{\partial M}{\partial \Omega} & 0 & 0 & 0 \\
\frac{\partial M}{\partial a} & \frac{\partial M}{\partial e} & \frac{\partial M}{\partial \omega} & 0 & 0 & 0 \\
\frac{\partial M}{\partial a} & \frac{\partial M}{\partial e} & \frac{\partial M}{\partial \Omega} & 0 & 0 & 0 \\
\frac{\partial M}{\partial a} & \frac{\partial M}{\partial e} & \frac{\partial M}{\partial \omega} & 0 & 0 & 0
\end{bmatrix}
$$

(2.55)

The $B$ matrix is made of the terms of the well-known Gauss Variational Equations (GVEs) [46]. The GVEs relate the impact of a perturbation or control acceleration on each of the orbit elements. Thus, the elements of the $B$ matrix are:

$$B = \begin{bmatrix}
\frac{2a_c^2e_c \sin \nu_c}{h_c} & \frac{2a_c^2p_c}{r_c} & 0 \\
p_c \sin \nu_c & (p_c + r_c) \cos r_c + r_c e_c & 0 \\
0 & h_c & r_c \cos (\nu_c + \omega_c) \\
0 & 0 & r_c \sin (\nu_c + \omega_c) \\
-\frac{p_c \cos \nu_c}{h_c e_c} & \frac{(p_c + r_c) \sin \nu_c}{h_c e_c} & 0 \\
\frac{b_c p_c \cos \nu - 2 r_c e_c}{a_c h_c e_c} & -\frac{(p_c + r_c) \sin \nu_c}{h_c e_c} & 0
\end{bmatrix}
$$

(2.56)

where $p_c = a_c (1 - e_c^2)$ is the semilatus rectum of the chief’s orbit, $h_c = \sqrt{\mu p_c}$ the reference orbit’s angular momentum (with $\mu$ being Earth’s gravitational parameter) and $b_c = a_c \sqrt{1 - e_c^2}$ the reference orbit’s semiminor axis. This linear time-varying, or more accurately “chief orbit element-varying”, state-space form will accurately model the secular relative drift caused by $J_2$ on an eccentric orbit, but fails to model the relative short-period oscillations between the chief and the deputy caused by the $J_2$ perturbation. Nevertheless, this linear model could prove to be useful in the design of control systems that make use of linear time-varying state-space models, such as model-predictive controllers or gain-scheduling controllers.
2.7 Simulation Results

The accuracy of the closed-form solution (Eq. 2.41 to 2.43 and Eq. 2.46 to 2.48) was evaluated with respect to the "true" relative dynamics, based on numerical integration of \( J_2 \)-perturbed dynamics. This accuracy is also compared with the accuracy of the elliptical linearized equations of unperturbed elliptical motion [27], referred to as the unperturbed elliptical motion model. The chief orbit elements \( e_0 \) were set to a slightly elliptical 45 deg inclined low-Earth orbit, as described in Table 2.1. The deputy was given a small orbit element offset \( \delta e_0 \) as shown in Table 2.2. Only \( \delta a_0 \) was set to 0. The reason is that in an unperturbed environment, this condition is sufficient to ensure non-drifting relative motion. However, in a \( J_2 \)-perturbed environment, the non-drifting conditions are slightly different [44]. Therefore, those conditions lead to a secular relative drift that a set of equations that does not include the \( J_2 \) perturbation will not be able to model. The resulting relative position in a \( J_2 \)-perturbed environment, obtained with numerical integration, is presented in Fig. 2.2, while Fig. 2.3 shows the resulting relative velocity.

Figure 2.4 shows the relative position error between the predicted and the true curvilinear Hill frame coordinates for both models (perturbed and unperturbed) for 10 orbital periods. As expected, the unperturbed elliptical motion model cannot predict the relative drift caused by the \( J_2 \) perturbation and shows a growing relative error in \( x \), \( y \) and \( z \). On the other hand, the model developed here, that includes the \( J_2 \) perturbation

<table>
<thead>
<tr>
<th>( e_0 )</th>
<th>( a_0 )</th>
<th>( e_0 )</th>
<th>( i_0 )</th>
<th>( \Omega_0 )</th>
<th>( \omega_0 )</th>
<th>( M_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.1RE</td>
<td>0.05</td>
<td>( \pi/4 )</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>
TABLE 2.2 Deputy Initial Orbit Element Offset

<table>
<thead>
<tr>
<th>$\delta e_0$</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta a_0$</td>
<td>0</td>
</tr>
<tr>
<td>$\delta e_0$</td>
<td>+0.0001</td>
</tr>
<tr>
<td>$\delta i_0$</td>
<td>+0.0001</td>
</tr>
<tr>
<td>$\delta \Omega_0$</td>
<td>-0.0001</td>
</tr>
<tr>
<td>$\delta \omega_0$</td>
<td>-0.0001</td>
</tr>
<tr>
<td>$\delta M_0$</td>
<td>+0.0001</td>
</tr>
</tbody>
</table>

through relative mean orbit element differences, does not show any secularly growing error and models accurately the long-term effect of $J_2$. The short-period variations of the position errors are mainly due to the neglected relative short-period motion between the chief and the deputy. This error is bounded within 1.5% of the maximum relative position at any point in time on all three axes. Therefore, only other perturbations, such as other gravitational harmonics, solar radiation pressure or atmospheric drag would affect the long-term accuracy of the model. With the linearized elliptical model, the error grows by approximately 1% per orbit, reaching nearly 13% after 10 orbits on the $x$, $y$ and $z$ axes. The same conclusions can be drawn for velocity (Fig. 2.3 and 2.5). Only short-period perturbations affect the $\dot{x}$, $\dot{y}$ and $\dot{z}$ errors.

### 2.8 Conclusion

A linearized analytical set of equations that provides the position and velocity of a deputy orbiting close to a chief on a $J_2$-perturbed elliptical orbit has been developed. This model provides a simpler state transition matrix for relative motion by assuming that relative motion in osculating coordinates is only caused by relative mean orbit element drift. The model uses the linearized drift rate difference of the mean orbit element difference to predict the relative secular drift caused by $J_2$. It has proven to model accurately the secular drift caused by $J_2$, leaving only errors caused by short-period relative motion between the chief and the deputy. It has been shown that even though simplifi-
Figure 2.2  In-plane and out-of-plane deputy relative motion in Hill frame for 10 orbits in a $J_2$-perturbed environment.
Figure 2.3 Relative velocity in Hill frame for 10 orbits in a \( J_2 \)-perturbed environment.

Cations have been made, only a bounded prediction error remains, even for long-term prediction. This error is bounded by the size of the relative motion induced by short-periodic \( J_2 \) perturbation terms.

The model provides the position and velocity of a deputy orbiting near a chief spacecraft for any point of the orbit only from the relative orbit elements vector and the initial state vector of the chief. The model avoids the need to numerically propagate forward in time the states of the chief. The model can be numerically inverted to yield the current required orbit element differences to reach a desired formation (described as position and velocity in Hill coordinates) at a specific true anomaly. The consideration of the \( J_2 \) perturbation in the model allows the use of the model several orbits in advance, because virtually no secular drift error caused by \( J_2 \) remains. Only errors caused by other perturbations, such as other gravitational field harmonics (\( J_3, J_4 \), etc.), solar radiation pressure or differential drag will affect the long-term modeling accuracy of the model. A linear time-varying state-space expression of the model has also been presented.

This model would be particularly well-suited for on-board guidance and control applications as the analytical equations require no numerical iteration to predict the relative motion once the initial mean and osculating orbit elements of the chief are known.
Figure 2.4 Position modelling error for the linearized elliptical motion model and the linearized elliptical motion considering the $J_2$ perturbation in a $J_2$-perturbed environment.
Figure 2.5  Velocity modelling error for the linearized elliptical motion model and the linearized elliptical motion considering the $J_2$ perturbation in a $J_2$-perturbed environment.
The naturally induced secular relative motion by the $J_2$ perturbation is included in the model and can therefore be used to perform maneuvers. Furthermore, once the analytically defined $\Phi$ matrix is known, the relative motion of any spacecraft of the formation can be obtained with only one matrix multiplication, $\Phi \delta e$. The same matrix $\Phi$ can be used for all the spacecraft of the formation, assuming the relative orbit elements are sufficiently small. The only information required is the relative orbit elements of the spacecraft in the formation. This approach would prove to be more and more efficient as the number of spacecraft of the formation becomes large.

In order to simplify the terms of the state transition matrix, classical orbit elements are used. Thus, the model can only be applied to non-circular and inclined orbits. For circular or equatorial orbits, some of the classical orbit elements used in this model are not defined. For both cases, previously existing models can be used or an extension to nonsingular orbit elements could also be implemented.

### 2.9 Appendix: Linearization of the Osculating to Mean Orbit Elements Mapping Function

The function [46]:

$$a = \xi_a = a - a \gamma_2 \left[ (3 \cos^2 i - 1) \left( \frac{a}{r} \right)^3 \frac{1}{\eta^3} + 3 \left( 1 - \cos^2 i \right) \left( \frac{a}{r} \right)^3 \cos (2\omega + 2\nu) \right]$$

where:

$$\gamma_2 = -\frac{J_2}{2} \left( \frac{P_e}{a} \right)^2$$

(2.57)

(2.58)

can be linearized about $e$ to provide an approximation of the mean semimajor axis increment from the osculating orbit element difference $\delta e$:

$$\delta a = \frac{\partial \xi_a}{\partial a} \delta a + \frac{\partial \xi_a}{\partial e} \delta e + \frac{\partial \xi_a}{\partial i} \delta i + \frac{\partial \xi_a}{\partial \omega} \delta \omega + \frac{\partial \xi_a}{\partial \nu} \left( \frac{(1 + e \cos \nu)^2}{\eta^3} \delta M + \frac{\sin \nu}{\eta^2} \delta e \right)$$

(2.59)
where \( \nu_0 \) is the initial true anomaly of the chief. The partial derivatives are:

\[
\frac{\partial \xi_a}{\partial a} = 1 - \gamma_2 \left[ (3 \cos^2 i - 1) \left( \frac{a}{r} \right)^3 - \frac{1}{\eta^3} \right] + 3 \left( 1 - \cos^2 i \right) \left( \frac{a}{r} \right)^3 \cos (2\omega + 2\nu) \\
\approx 1
\]

\[
\frac{\partial \xi_a}{\partial e} = a \gamma_2 \left[ (2 - 3 (\sin i)^2) \left( \frac{3 (1 + e \cos \nu)^2 \cos \nu}{\eta^6} + 6 \frac{(1 + e \cos \nu)^3 e}{\eta^6} - 3 \frac{e}{\eta^5} \right) \\
+9 \frac{\sin^2 i (1 + e \cos \nu)^2 \cos (2\omega + 2\nu) \cos \nu}{\eta^6} \\
+18 \frac{\sin^2 i (1 + e \cos \nu)^3 \cos (2\omega + 2\nu) e}{\eta^6} \right] \tag{2.60}
\]

\[
\frac{\partial \xi_a}{\partial i} = -3a \gamma_2 \sin (2i) \left[ \left( \frac{a}{r} \right)^3 (1 - \cos (2\omega + 2\nu)) - \frac{1}{\eta^3} \right] \tag{2.61}
\]

\[
\frac{\partial \xi_a}{\partial \omega} = -6a \gamma_2 \left( 1 - \cos^2 i \right) \frac{(a/r)^3}{\eta^5} \sin (2\omega + 2\nu) \tag{2.62}
\]

\[
\frac{\partial \xi_a}{\partial \nu} = a \gamma_2 \frac{(1 + e \cos \nu)^2}{\eta^6} \left[ (-9 \cos^2 i + 3) e \sin \nu \\
- (9 - 9 \cos^2 i) \cos (2\omega + 2\nu) e \sin \nu \\
- (6 - 6 \cos^2 i) (1 + e \cos \nu) \sin (2\omega + 2\nu) \right] \tag{2.63}
\]

\[ 2.10 \text{ Appendix: } \Phi \text{ Matrix} \]

This section presents the elements of a matrix \( \Phi \) defined as

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z} \\
x \\
y \\
z
\end{bmatrix} = \Phi(e_0, \nu) \begin{bmatrix}
\delta a_0 \\
\delta e_0 \\
\delta i_0 \\
\delta \Omega_0 \\
\delta \omega_0 \\
\delta M_0
\end{bmatrix}
\]

where \( e_0 \) is the initial orbit element vector of the chief and \( \nu \) the true anomaly for which the formation is needed. The elements of \( \Phi \) are obtained by collecting Eq. 2.41 to 2.43 and 2.46 to 2.48 and by using the definition of mean orbit element drift of Eq. 2.16 to 2.18, the partial derivatives of Eq. 2.22 to 2.30 and of Appendix 2.9 and the estimated flight time \( \tau \) of Eq. 2.40.
The elements of $\Phi$ for the computation of $\dot{x}$ are:

\[
\kappa_{x} = \frac{a_0 e_0 \dot{\nu} \cos \nu}{\eta_0}
\]

\[
\Phi_{11} = \frac{\dot{r}}{a_0} + \left( \kappa_{x_0} + \frac{a_0 e_0 \sin \nu}{\eta_0} \right) \frac{\partial \dot{M}}{\partial a}
\]

\[
\Phi_{12} = \left( \kappa_{x_0} + \frac{a_0 e_0 \sin \nu}{\eta_0} \right) \left[ \frac{\partial \dot{M}}{\partial e} \frac{\partial \xi_a}{\partial e} + \frac{\partial \dot{M}}{\partial \nu} \frac{\partial \xi_a}{\partial \nu} \frac{\nu_0^2}{\eta_0^2} (2 + e_0 \cos e_0) \right]
\]

\[
\Phi_{13} = \left( \kappa_{x_0} + \frac{a_0 e_0 \sin \nu}{\eta_0} \right) \left( \frac{\partial \dot{M}}{\partial i} \frac{\partial \xi_a}{\partial i} \right)
\]

\[
\Phi_{14} = 0
\]

\[
\Phi_{15} = \left( \kappa_{x_0} + \frac{a_0 e_0 \sin \nu}{\eta_0} \right) \frac{\partial \dot{M}}{\partial \omega} \frac{\partial \xi_a}{\partial \omega}
\]

\[
\Phi_{16} = \kappa_{x} + \left( \kappa_{x_0} + \frac{a_0 e_0 \sin \nu}{\eta_0} \right) \frac{\partial \dot{M}}{\partial a} \frac{\partial \xi_a (1 + e_0 \cos \nu_0)^2}{\nu_0^3 \eta_0^3}
\]
The elements of $\Phi$ for the computation of $\dot{y}$ are:

$$
\kappa_y = \frac{\dot{r} (1 + \epsilon_0 \cos \nu)^2}{\eta_0^3} - \frac{2\epsilon_0 \dot{\nu} (1 + \epsilon_0 \cos \nu) \sin \nu}{\eta_0^3}
$$

$$
\Phi_{21} = (\dot{r} \cos i_0 \tau + r \cos i_0) \frac{\partial \dot{\Omega}}{\partial \tilde{a}} + (\dot{\tau} + r) \frac{\partial \dot{\omega}}{\partial \tilde{a}} + \left[ \kappa_y \tau + \frac{r (1 + \epsilon_0 \cos \nu)^2}{\eta_0^3} \right] \frac{\partial \dot{M}}{\partial \tilde{a}}
$$

$$
\Phi_{22} = \frac{1}{\eta_0^3} \left[ r \dot{\nu} \cos \nu (2 + \epsilon_0 \cos \nu) - r \epsilon_0 \dot{\nu} \sin^2 \nu + \dot{r} \sin \nu (2 + \epsilon_0 \cos \nu) \right]
+ (\dot{r} \cos i_0 \tau + r \cos i_0) \left[ \frac{\partial \dot{\Omega}}{\partial \tilde{a}} \frac{\partial \xi_a}{\partial \theta} + \frac{\partial \dot{\xi}_a}{\partial \tilde{a}} \frac{\partial \dot{\nu}}{\partial \theta} \right] \left( 2 + \epsilon_0 \cos e_0 \right) + \frac{\partial \dot{\omega}}{\partial \theta}
+ \left[ \kappa_y \tau + \frac{r (1 + \epsilon_0 \cos \nu)^2}{\eta_0^3} \right] \left[ \frac{\partial \dot{M}}{\partial \tilde{a}} \frac{\partial \xi_a}{\partial \theta} + \frac{\partial \dot{M}}{\partial \tilde{a}} \frac{\partial \xi_a}{\partial \theta} \right] \left( 2 + \epsilon_0 \cos e_0 \right) + \frac{\partial \dot{M}}{\partial \theta}
$$

$$
\Phi_{23} = (\dot{r} \cos i_0 \tau + r \cos i_0) \left( \frac{\partial \dot{\Omega}}{\partial \tilde{a}} \frac{\partial \xi_a}{\partial \theta} + \frac{\partial \dot{\xi}_a}{\partial \tilde{a}} \frac{\partial \dot{\nu}}{\partial \theta} \right) + (\dot{\tau} + r) \left( \frac{\partial \dot{\omega}}{\partial \theta} \frac{\partial \xi_a}{\partial \theta} + \frac{\partial \dot{\xi}_a}{\partial \theta} \frac{\partial \dot{\nu}}{\partial \theta} \right)
+ \left[ \kappa_y \tau + \frac{r (1 + \epsilon_0 \cos \nu)^2}{\eta_0^3} \right] \left( \frac{\partial \dot{M}}{\partial \tilde{a}} \frac{\partial \xi_a}{\partial \theta} + \frac{\partial \dot{M}}{\partial \tilde{a}} \frac{\partial \xi_a}{\partial \theta} \right)
$$

$$
\Phi_{24} = \dot{r} \cos i_0
$$

$$
\Phi_{25} = \dot{r} + (\dot{r} \cos i_0 \tau + r \cos i_0) \frac{\partial \dot{\Omega}}{\partial \tilde{a}} \frac{\partial \xi_a}{\partial \theta} + (\dot{\tau} + r) \frac{\partial \dot{\xi}_a}{\partial \tilde{a}} \frac{\partial \dot{\nu}}{\partial \theta}
+ \left[ \kappa_y \tau + \frac{r (1 + \epsilon_0 \cos \nu)^2}{\eta_0^3} \right] \frac{\partial \dot{M}}{\partial \tilde{a}} \frac{\partial \xi_a}{\partial \theta}
$$

$$
\Phi_{26} = \kappa_y + (\dot{r} \cos i_0 \tau + r \cos i_0) \frac{\partial \dot{\Omega}}{\partial \tilde{a}} \frac{\partial \xi_a (1 + \epsilon_0 \cos \nu)^2}{\partial \theta} \frac{\eta_0^3}{\partial \theta}
+ (\dot{\tau} + r) \frac{\partial \dot{\xi}_a (1 + \epsilon_0 \cos \nu)^2}{\partial \tilde{a}} \frac{\eta_0^3}{\partial \theta}
+ \left[ \kappa_y \tau + \frac{r (1 + \epsilon_0 \cos \nu)^2}{\eta_0^3} \right] \frac{\partial \dot{M}}{\partial \tilde{a}} \frac{\partial \xi_a (1 + \epsilon_0 \cos \nu)^2}{\partial \theta} \frac{\eta_0^3}{\partial \theta}
$$
The elements of $\Phi$ for the computation of $\dot{z}$ are:

\[ \kappa_{z} = -\dot{r} \cos(\nu + \omega_0 + \dot{\omega} \tau) \sin i_0 + r \sin(\nu + \omega_0 + \dot{\omega} \tau) (\dot{\nu} + \dot{\omega}) \sin i_0 \]

\[ \Phi_{31} = [\kappa_{z} \tau - r \cos(\nu + \omega_0 + \dot{\omega} \tau) \sin i_0] \frac{\partial \dot{\Omega}}{\partial \tau} \]

\[ \Phi_{32} = [\kappa_{z} \tau - r \cos(\nu + \omega_0 + \dot{\omega} \tau) \sin i_0] \left[ \frac{\partial \dot{\Omega}}{\partial \dot{e}} \frac{\partial \xi_a}{\partial e} + \frac{\partial \dot{\Omega}}{\partial \dot{a}} \frac{\partial \xi_a}{\partial \nu} \frac{\nu_0^2 (2 + e_0 \cos e_0)}{\eta_0^3} + \frac{\partial \dot{\Omega}}{\partial \dot{\omega}} \right] \]

\[ \Phi_{33} = \dot{r} \sin(\nu + \omega_0 + \dot{\omega} \tau) + r \cos(\nu + \omega_0 + \dot{\omega} \tau) (\dot{\nu} + \dot{\omega}) \]

\[ + [\kappa_{z} \tau - r \cos(\nu + \omega_0 + \dot{\omega} \tau) \sin i_0] \frac{\partial \dot{\Omega}}{\partial \dot{a}} \]

\[ \Phi_{34} = \kappa_{z} \]

\[ \Phi_{35} = [\kappa_{z} \tau - r \cos(\nu + \omega_0 + \dot{\omega} \tau) \sin i_0] \frac{\partial \dot{\Omega}}{\partial \dot{a}} \frac{\partial \xi_a}{\partial \dot{a}} \frac{\partial \xi_a}{\partial \omega} \]

\[ \Phi_{36} = [\kappa_{z} \tau - r \cos(\nu + \omega_0 + \dot{\omega} \tau) \sin i_0] \frac{\partial \dot{\Omega}}{\partial \dot{a}} \frac{\partial \xi_a}{\partial \nu} \frac{(1 + e_0 \cos e_0)^2}{\eta_0^3} \]

The elements of $\Phi$ for the computation of $x$ are:

\[ \Phi_{41} = \frac{r}{a_0} + \frac{a_0 e_0 \sin \nu}{\eta_0} \frac{\partial \dot{M}}{\partial \dot{a}} \tau \]

\[ \Phi_{42} = \frac{a_0 e_0 \sin \nu}{\eta_0} \left[ \frac{\partial \dot{M}}{\partial \dot{a}} \frac{\partial \xi_a}{\partial e} + \frac{\partial \dot{M}}{\partial \dot{a}} \frac{\partial \xi_a}{\partial \nu} \frac{\nu_0^2 (2 + e_0 \cos e_0)}{\eta_0^3} + \frac{\partial \dot{M}}{\partial \dot{\omega}} \right] \tau - a_0 \cos \nu \]

\[ \Phi_{43} = \frac{a_0 e_0 \sin \nu}{\eta_0} \left( \frac{\partial \dot{M}}{\partial \dot{a}} \frac{\partial \xi_a}{\partial \dot{a}} + \frac{\partial \dot{M}}{\partial \dot{a}} \right) \tau \]

\[ \Phi_{44} = 0 \]

\[ \Phi_{45} = \frac{a_0 e_0 \sin \nu}{\eta_0} \left( \frac{\partial \dot{M}}{\partial \dot{a}} \frac{\partial \xi_a}{\partial \dot{a}} \frac{\partial \xi_a}{\partial \omega} \right) \tau \]

\[ \Phi_{46} = \frac{a_0 e_0 \sin \nu}{\eta_0} + \frac{a_0 e_0 \sin \nu}{\eta_0} \frac{\partial \dot{M}}{\partial \dot{a}} \frac{\partial \xi_a}{\partial \omega} \frac{(1 + e_0 \cos \nu_0)^2}{\eta_0^3} \tau \]
The elements of $\Phi$ for the computation of $y$ are:

$$
\Phi_{51} = r \cos i_0 \frac{\partial \dot{\Omega}}{\partial a} \tau + r \cos i_0 \frac{\partial \dot{\omega}}{\partial a} \tau + \frac{r(1 + e_0 \cos \nu_0)^2}{\eta_0^3} \frac{\partial \dot{M}}{\partial a} \tau $$

$$
\Phi_{52} = -\frac{r \sin \nu}{\eta_0^2} (2 + e_0 \cos \nu) + r \cos i_0 \left[ \frac{\partial \dot{\Omega}}{\partial \omega} \frac{\partial \xi_a}{\partial \omega} + \frac{\partial \dot{\Omega}}{\partial \nu} \frac{\partial \xi_a}{\partial \nu} \frac{\sin \nu_0}{\eta_0^2} (2 + e_0 \cos \nu) + \frac{\partial \dot{\Omega}}{\partial \nu} \right] \tau 

+ \frac{r(1 + e_0 \cos \nu_0)^2}{\eta_0^3} \frac{\partial \dot{M}}{\partial a} \frac{\partial \xi_a}{\partial \omega} \sin \nu_0 \frac{\eta_0^3}{\eta_0^2} (2 + e_0 \cos \nu) + \frac{\partial \dot{M}}{\partial \nu} \tau 

\Phi_{53} = r \cos i_0 \left( \frac{\partial \dot{\Omega}}{\partial a} \frac{\partial \xi_a}{\partial \dot{a}} + \frac{\partial \dot{\Omega}}{\partial \dot{a}} \right) \tau + r \left( \frac{\partial \dot{\Omega}}{\partial \omega} \frac{\partial \xi_a}{\partial \dot{a}} + \frac{\partial \dot{\Omega}}{\partial \nu} \right) \tau 

+ \frac{r(1 + e_0 \cos \nu_0)^2}{\eta_0^3} \left( \frac{\partial \dot{M}}{\partial a} \frac{\partial \xi_a}{\partial \dot{a}} + \frac{\partial \dot{M}}{\partial \dot{a}} \right) \tau 

\Phi_{54} = r \cos i_0 

\Phi_{55} = r + r \cos i_0 \frac{\partial \dot{\Omega}}{\partial a} \tau + r \frac{\partial \dot{\omega}}{\partial a} \tau + \frac{r(1 + e_0 \cos \nu_0)^2}{\eta_0^3} \frac{\partial \dot{M}}{\partial a} \tau 

\Phi_{56} = \frac{r(1 + e_0 \cos \nu)^2}{\eta_0^3} \frac{r \cos i_0}{\eta_0^3} \frac{\partial \dot{\Omega}}{\partial a} \frac{\partial \xi_a}{\partial \dot{a}} \tau + \frac{r \cos i_0}{\eta_0^3} \frac{\partial \dot{\Omega}}{\partial \nu} \frac{\partial \xi_a}{\partial \nu} \tau 

+ r \frac{r(1 + e_0 \cos \nu)^2}{\eta_0^3} \frac{\partial \dot{M}}{\partial a} \frac{\partial \xi_a}{\partial \dot{a}} \tau 

The elements of $\Phi$ for the computation of $z$ are:

$$
\Phi_{61} = -r \cos(\nu + \omega + \dot{\omega}) \sin i_0 \frac{\partial \dot{\Omega}}{\partial a} \tau 

\Phi_{62} = -r \cos(\nu + \omega + \dot{\omega}) \sin i_0 \left[ \frac{\partial \dot{\Omega}}{\partial \omega} \frac{\partial \xi_a}{\partial \omega} + \frac{\partial \dot{\Omega}}{\partial \nu} \frac{\partial \xi_a}{\partial \nu} \frac{\sin \nu_0}{\eta_0^2} (2 + e_0 \cos \nu) + \frac{\partial \dot{\Omega}}{\partial \nu} \right] \tau 

\Phi_{63} = r \sin(\nu + \omega + \dot{\omega}) - r \cos(\nu + \omega + \dot{\omega}) \sin i_0 \left( \frac{\partial \dot{\Omega}}{\partial a} \frac{\partial \xi_a}{\partial a} + \frac{\partial \dot{\Omega}}{\partial \dot{a}} \frac{\partial \xi_a}{\partial \dot{a}} \right) \tau 

\Phi_{64} = -r \cos(\nu + \omega + \dot{\omega}) \sin i_0 

\Phi_{65} = -r \cos(\nu + \omega + \dot{\omega}) \sin i_0 \frac{\partial \dot{\Omega}}{\partial a} \frac{\partial \xi_a}{\partial \omega} \tau 

\Phi_{66} = -r \cos(\nu + \omega + \dot{\omega}) \sin i_0 \frac{\partial \dot{\Omega}}{\partial a} \frac{\partial \xi_a}{\partial \nu} \frac{(1 + e_0 \cos \nu_0)^2}{\eta_0^3} \tau
CHAPTER 3

Fuel-equivalent Relative Orbit Element Space


J. Hamel and J. de Lafontaine, *AIAA Journal of Guidance, Control, and Dynamics*, accepted for publication [22].

Abstract

This paper presents a new tool to analytically perform the guidance for reconfiguration of formation flying spacecraft. The technique consists in mapping the relative orbit elements into a fuel-equivalent space where similar displacements correspond to an equivalent fuel consumption. The minimal-fuel maneuver problem is consequently translated into a simple geometric problem in the fuel-equivalent space. The theory is applied to two well-known formations: the $J_2$-invariant formation and the Projected Circular Formation. The use of the fuel-equivalent space leads to very simple solutions for the most fuel-efficient way to attain both formations.

3.1 Introduction

There has been undoubtedly a paradigm shift in the last years toward the use of spacecraft formation flying. Formation flying replaces large and expensive spacecraft by several smaller spacecraft that can perform the same mission with an increased reconfigurability and robustness to failures. However, formation flying increases the
complexity of some systems, mainly of the guidance, navigation and control functions, which rapidly grow in complexity with the number of spacecraft in the formation.

This conflicts however with the increasing need for autonomy to decrease the cost of ground support. Ground support operations are still a non-negligible part of the cost of a mission, especially for small scientific missions with small budgets. This naturally leads to a need for more autonomous guidance, navigation and control algorithms that can perform autonomous decisions and trade-offs that would otherwise be performed by the ground segment. This also becomes very challenging for formations with a large number of spacecraft.

This paper therefore concentrates on the development of a new tool to autonomously perform formation flying guidance. The purpose of the guidance system is to provide a reference trajectory to reach a specific formation. This trajectory can optimize the duration of the maneuver, optimize the fuel cost of the maneuver, minimize the risk of collision or do all three at the same time.

The most common approach in formation flying guidance is the use of computationally expensive techniques. Such examples are the use of linear programming [51, 35], multi-agent optimization techniques [57], particle swarm optimization [24], genetic algorithms [2] or optimal control theory [9, 55]. These kinds of techniques have a lot of freedom in the selection of the quantity to optimize and the constraints to impose. They all provide ways to compute the best maneuver to reach a desired formation. However, these methods require an initially unknown (and most likely large) number of iterations and convergence is not always guaranteed. Obviously, this precludes any on-board implementation of this type of algorithm.

On the other hand, analytical solutions to the optimal reconfiguration problem can be found under certain conditions. Indeed, unperturbed circular reference orbits lead to simple analytical expressions and easily expressed configurations [40]. Mishne [33] almost analytically solves the optimal control problem for circular orbits for power-limited thrusters (only a small amount of numerical optimization remains). Further-
more, Vaddi [53] developed an analytical and simple solution to the circular formation establishment and reconfiguration using impulsive thrusters about a circular reference orbit. On the other hand, Gurfil [14] proposes an analytical and optimal way of reaching bounded relative motion for any Keplerian orbits with only one impulse through the application of an energy-matching constraint. However, even though this impulse guarantees orbit-periodic relative motion, it is not made to aim for a specific configuration.

It is the intent of this paper to propose a new tool that yields an analytical solution to the fuel-optimal reconfiguration problem for any type of orbit for any geometrically simple formation. To do so, spacecraft relative position and desired formations are mapped into a "fuel-equivalent" space where equivalent distances on all axes relate to identical fuel consumption. This mapping is a way to rapidly compute the most fuel-efficient way to reach a formation by taking the shortest path in the fuel-equivalent space, reducing the problem to a simple geometric problem. It avoids the need to perform a systematic search as is traditionally done [35].

Section 3.2 first reviews the impulsive feedback controller [43, 46], upon which the fuel-equivalent space theory is built. Then, section 3.3 defines the fuel-equivalent space. Finally, sections 3.4 and 3.5 provide two examples of how this theory can be applied to compute the most fuel-efficient maneuvers for two well known formation flying cases: the $J_2$-invariant relative orbits and the Projected Circular Formation (PCF).

### 3.2 Impulsive Feedback Controller

The impulsive feedback controller [43, 46] was proposed as a way to perform orbit element corrections while minimizing the impact on the other orbit elements. It is based on the Gauss Variational Equations and can perform any arbitrary small orbit correction with only three impulses. If only one or two elements are to be corrected, the controller provides essentially optimal results in terms of fuel. If all six elements are to be corrected, the controller proposes maneuvers that are only a few percents larger
CHAPTER 3. FUEL-EQUIVALENT RELATIVE ORBIT ELEMENT SPACE

than the optimal multi-impulse solution. However, the most important advantage of this technique is that the impulses and their location can be computed analytically with very simple expressions, leading very quickly to a good approximation of the fuel-cost of a maneuver, even if the spacecraft does not make use of impulsive thrusters.

The controller performs the corrections $\Delta e = [\Delta a \; \Delta e \; \Delta i \; \Delta \Omega \; \Delta \omega \; \Delta M]^T$ on all six orbit elements (i.e. the semimajor axis $a$, the eccentricity $e$, the inclination $i$, the right ascension of the ascending node $\Omega$, the argument of periapsis $\omega$ and the mean anomaly $M$) with only three impulses $\Delta v_p$, $\Delta v_a$ and $\Delta v_h$ respectively at the periapsis, at the apoapsis and at a critical true latitude angle $\theta_c$.

The first impulse, $\Delta v_h$ performs both inclination and ascending node corrections in one single normal impulse. Obviously it is more efficient to correct the inclination when the spacecraft crosses equator and to correct ascending node near the poles, but it is more fuel-efficient to correct both elements with one single impulse if both have to be corrected. This normal impulse is to take place at the critical true latitude angle $\theta_c$:

$$\theta_c = \arctan \frac{\Delta \Omega \sin i}{\Delta i}$$  \hspace{1cm} (3.1)

and its magnitude is:

$$\Delta v_h = \sqrt{(\Delta v_{hi})^2 + (\Delta v_{h\Omega})^2}$$  \hspace{1cm} (3.2)

where

$$\Delta v_{hi} = \frac{h}{r} \Delta i$$  \hspace{1cm} (3.3)

$$\Delta v_{h\Omega} = \frac{h}{r} \Delta \Omega \sin i$$  \hspace{1cm} (3.4)

and where $h$ is the orbit angular momentum and $r$ the orbit equatorial radius. This normal impulse has an impact on $i$, $\Omega$ and $\omega$. Therefore, this effect on $\omega$ is compensated through another impulse.

The argument of periapsis and the mean anomaly are also corrected as a pair, but through two radial impulses. Those two impulses are to take place at apoapsis ($\Delta v_{ra}$)
and at periapsis ($\Delta v_{r_p}$). The magnitudes of the radial impulses are:

$$
\Delta v_{r_p} = \frac{na}{4} \left[ \frac{(1 + e)^2}{\eta} \left( \Delta \omega + \Delta \Omega \cos i \right) + \Delta M \right] \quad (3.5)
$$

$$
\Delta v_{r_a} = \frac{na}{4} \left[ \frac{(1 - e)^2}{\eta} \left( \Delta \omega + \Delta \Omega \cos i \right) + \Delta M \right] \quad (3.6)
$$

where $n = \sqrt{\mu/a^3}$ is the orbit mean motion, $\mu$ the gravitational parameter and $\eta = \sqrt{1 - e^2}$. The remaining two tangential impulses are used to correct the orbit semimajor axis and the eccentricity. The two impulses are once again performed at periapsis and at apoapsis:

$$
\Delta v_{t_p} = \frac{n\eta}{4} \left( \frac{\Delta a}{a} + \frac{\Delta e}{1 + e} \right) \quad (3.7)
$$

$$
\Delta v_{t_a} = \frac{n\eta}{4} \left( \frac{\Delta a}{a} - \frac{\Delta e}{1 - e} \right) \quad (3.8)
$$

Therefore, the near-optimal fuel cost of a small relative orbit element correction $\Delta e$ can easily be estimated with the results of the impulsive feedback controller:

$$
\Delta v = \sqrt{(\Delta v_{r_a})^2 + (\Delta v_{t_a})^2 + (\Delta v_{r_p})^2 + (\Delta v_{t_p})^2 + (\Delta v_{h_\alpha})^2 + (\Delta v_{h_i})^2} \quad (3.9)
$$

This result is used to translate orbit element errors directly into fuel cost. It is used next to map these errors into the fuel-equivalent space.

### 3.3 Fuel-Equivalent Space

The relative orbit elements can be translated into six fuel-equivalent coordinates:

$$
\delta V = \left[ \delta V_{t_p} \delta V_{t_a} \delta V_{h_\alpha} \delta V_{h_\alpha} \delta V_{r_p} \delta V_{r_a} \right]^T \quad (3.10)
$$

The $\delta V_{t_p}$, $\delta V_{t_a}$, $\delta V_{r_p}$ and $\delta V_{r_a}$ coordinates represent the magnitude of the tangential (Eq. 3.7 and 3.8) and radial (Eq. 3.5 and 3.6) components of the apoapsis and periapsis impulses required to perform the $\Delta a$, $\Delta e$, $\Delta \Omega$, $\Delta \omega$ and $\Delta M$ corrections that would take the spacecraft from the origin to its current location. In turn, the $\delta V_{h_i}$ and $\delta V_{h_\alpha}$ coordinates represent the impact on the magnitude of the normal impulse of $\Delta i$ and
ΔΩ corrections (Eq. 3.2). In a formation flying context, the origin of the fuel-equivalent space would be the reference trajectory of the formation, or the "leader". Thus, the coordinates of all the elements of the formation would be mapped in the same fuel-equivalent space with the same origin.

This linear mapping is thus performed through:

$$\delta V = S \delta e$$

(3.11)

where the mapping matrix $S$ is:

$$S = \begin{bmatrix}
\frac{n\eta}{4} & \frac{na\eta}{4(1+e)} & 0 & 0 & 0 & 0 \\
\frac{n\eta}{4} & \frac{na\eta}{4(1-e)} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{h}{r} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{h\sin i}{r} & 0 & 0 \\
0 & 0 & 0 & -\frac{na}{4\eta} (1+e)^2 \cos i & -\frac{na}{4\eta} (1+e)^2 & -\frac{na}{4} \\
0 & 0 & 0 & -\frac{na}{4\eta} (1-e)^2 \cos i & -\frac{na}{4\eta} (1-e)^2 & -\frac{na}{4} \\
\end{bmatrix}$$

(3.12)

and $\delta e$ is the relative orbit element vector with respect to the reference trajectory of the formation. This transformation translates the relative orbit elements into a six-dimensional space where the same displacement on any axis leads to an identical fuel cost. Thus, minimizing the distance in the fuel-equivalent space minimizes the fuel cost of a maneuver.

However, the true distance, in terms of fuel, between two points in the fuel-equivalent space is not the commonly-used Euclidean norm. The use of the traditional Euclidean norm to compute distance leads to an underestimation of the fuel cost, because the $\Delta v_a$, $\Delta v_p$, and $\Delta v_h$ corrections cannot be performed at the same time (they have to be performed at different locations of the orbit). Therefore, simultaneous displacements in the $\delta V_{r_a}, \delta V_{t_a}, \delta V_{r_p}, \delta V_{t_p}$ and the $\delta V_{h_i}, \delta V_{h_o}$ planes are not allowed.
3.4. EXAMPLE OF THE $J_2$-INVARIANT RELATIVE ORBITS

More rigorously, the fuel-equivalent distance $d_{fe}$ between two points $\delta V_1$ and $\delta V_2$, in the 6-dimensional fuel-equivalent space is defined as:

$$d_{fe} = \sqrt{(\delta V_{r_2} - \delta V_{r_1})^2 + (\delta V_{r_3} - \delta V_{r_3})^2 + (\delta V_{r_5} - \delta V_{r_5})^2 + (\delta V_{r_7} - \delta V_{r_7})^2 + (\delta V_{h_2} - \delta V_{h_2})^2 + (\delta V_{h_3} - \delta V_{h_3})^2} \quad (3.13)$$

The distance $d_{fe}$ provides an estimation of fuel cost of the optimal value of maneuvering from $\delta V_1$ to $\delta V_2$, with an accuracy similar to the results of the impulsive feedback controller.

This mapping thus translates the computation of minimal fuel-cost into a geometric problem in the fuel-equivalent space. When desired formations can be described geometrically, systematic search of the optimal solution can be replaced by an analytical solution obtained by studying the geometry of the problem. This can be applied, for example to the $J_2$-invariant orbits and the Projected Circular Formation.

3.4 Example of the $J_2$-invariant Relative Orbits

This section shows how the fuel-equivalent space theory can be used to obtain a simple analytical solution to finding the closest (in terms of fuel) $J_2$-invariant relative orbit.

To a first-order approximation, the $J_2$-invariant conditions enforce the relative secular drift caused by $J_2$-perturbations of all elements to be 0, except for the argument of periapsis and the mean anomaly for which the sum of the relative mean-element drift rates will be 0.

$J_2$-invariant orbits are defined by [42, 46]:

$$\delta a = -\frac{2 Da e}{\eta} \delta e \quad (3.14)$$

$$\delta e = \frac{(1 - e^2) \tan i}{4e} \delta i \quad (3.15)$$
where:

\[
D = \frac{J_2}{4L^4\eta^5} (4 + 3\eta)(1 + 5\cos^2 i) \quad (3.16)
\]

\[
L = \sqrt{a/R_e} \quad (3.17)
\]

and where \(\delta a\), \(\delta e\) and \(\delta i\) are the relative semi-major axis, eccentricity and inclination of a deputy with respect to a chief (or a reference trajectory) and \(R_e\) is the planet's equatorial radius. This means that the admissible \(J_2\)-invariant relative orbits are defined by two linear constraints on the selection of \(\delta a\), \(\delta e\) and \(\delta i\) (Eq. 3.14 and 3.15), which can be graphically represented as a straight line crossing the origin in the \(\delta a-\delta e-\delta i\) space. Because the mapping between the relative orbit element space and the fuel-equivalent space is linear, the \(J_2\)-invariant subset is also a straight line in the fuel-equivalent space.

Because only \(\delta a\), \(\delta e\) and \(\delta i\) corrections will be required, only 3 of the 6 dimensions are relevant and the mapping of this problem into the fuel-equivalent space takes a very simple form:

\[
\begin{bmatrix}
\delta V_{t_p} \\
\delta V_{t_a} \\
\delta V_{h_i}
\end{bmatrix} = S
\begin{bmatrix}
\delta a \\
\delta e \\
\delta i
\end{bmatrix}
\]

so that the distance \(d_{fe}\) between two points \(\delta V_1\) and \(\delta V_2\) is determined by:

\[
d_{fe} = |\delta V_{t_p2} - \delta V_{t_p1}| + |\delta V_{t_a2} - \delta V_{t_a1}| + |\delta V_{h_i2} - \delta V_{h_i1}|
\]

Therefore, finding the closest (in terms of fuel) \(J_2\)-invariant relative orbit from an initial relative orbit \(\delta e_0\) reduces to finding the location on a straight line (the \(J_2\)-invariant subset) that is the closest to a point (the initial spacecraft location), but with the distance defined as the sum of the absolute value of all the elements of the relative position vector.

The problem consists of finding the coordinates of the closest \(J_2\)-invariant relative orbit \(\delta V_2\) from the current coordinates \(\delta V_1\). Because the \(J_2\)-invariant subset is a straight line,
\[ \delta V_2 \text{ needs to satisfy } \delta V_2 = \delta V_A + u (\delta V_B - \delta V_A) \text{ where } \delta V_B \text{ and } \delta V_A \text{ can be any two arbitrary but distinct } J_2\text{-invariant coordinates (satisfying Eq. 3.14 and 3.15) and } u \text{ is a scalar.} \]

If this constraint is enforced, the distance \( d_{fe} \) between \( \delta V_1 \) and \( \delta V_2 \) can be expressed as:

\[
d_{fe} = |\delta V_{tp_1} - \delta V_{tp_A} - u (\delta V_{tp_B} - \delta V_{tp_A})| + |\delta V_{ta_1} - \delta V_{ta_A} - u (\delta V_{ta_B} - \delta V_{ta_A})| + |\delta V_{hi_1} - \delta V_{hi_A} - u (\delta V_{hi_B} - \delta V_{hi_A})| \tag{3.20}
\]

or more simply:

\[
d_{fe} = |d_{tp}| + |d_{ta}| + |d_{hi}| \tag{3.21}
\]

We seek the value of \( u \) that will minimize \( d_{fe} \). Obviously, one of the ways of doing so is by studying the value of the derivative of \( d_{fe} \) with respect to \( u \):

\[
\frac{d}{du} d_{fe} = -\frac{|d_{tp}|}{d_{tp}} (\delta V_{tp_B} - \delta V_{tp_A}) - \frac{|d_{ta}|}{d_{ta}} (\delta V_{ta_B} - \delta V_{ta_A}) - \frac{|d_{hi}|}{d_{hi}} (\delta V_{hi_B} - \delta V_{hi_A}) \tag{3.22}
\]

The expression of the derivative of \( d_{fe} \) reveals that \( d_{fe} \) is a linear function of \( u \) between singularities. These singularities will happen when the derivative of \( d_{fe} \) is undefined, \( i.e. \) when \( d_{tp}, d_{ta} \) or \( d_{hi} \) is 0. Thus, it can be graphically represented as four line segments linked by three singularities.

Furthermore, because

\[
\lim_{u \to -\infty} d_{fe} = \infty \tag{3.23}
\]

and

\[
\lim_{u \to -\infty} d_{fe} = \infty \tag{3.24}
\]

and because \( d_{fe} \) is linear between singularities, it can be shown that a minimum value for \( d_{fe} \) does exist and is inevitably found at one of the singularities. Indeed, the derivative of \( d_{fe} \) is a constant at every other location. Even if the derivative of \( d_{fe} \) is zero for
a given line segment, the singularities found at the boundary of this line segment are still at the same distance as the complete line segment.

This leads to the convenient conclusion that the closest \( J_2 \)-invariant orbit (in terms of fuel) will be located at \( \delta V_{t_{p_2}} = \delta V_{t_{p_1}} \), \( \delta V_{t_{a_2}} = \delta V_{t_{a_3}} \), or \( \delta V_{h_{k_2}} = \delta V_{h_{k_3}} \). The combination of this result with the conditions for \( J_2 \)-invariance leads to three potential maneuvers, one of which will be the most fuel-effective way to reach a \( J_2 \)-invariant orbit.

The first case is \( \delta V_{t_{p_2}} = \delta V_{t_{p_1}} \). This means that no tangential impulse is performed at periapsis to reach the \( J_2 \)-invariant orbit. This yields the condition:

\[
\Delta V_{t_p} = \frac{n\eta}{4} \left( \frac{\Delta a}{a} + \frac{\Delta e}{1 + e} \right) = 0
\]  

(3.25)

where the corrections \( \Delta a \) and \( \Delta e \) are the maneuvers required to reach the \( J_2 \)-invariance conditions \( \delta a_{\text{inv}}, \delta e_{\text{inv}} \) and \( \delta i_{\text{inv}} \) from the current coordinates \( \delta a_0, \delta e_0 \) and \( \delta i_0 \):

\[
\Delta a = \delta a_0 - \delta a_{\text{inv}} \\
\Delta e = \delta e_0 - \delta e_{\text{inv}} \\
\Delta i = \delta i_0 - \delta i_{\text{inv}}
\]

(3.26) (3.27) (3.28)

However, the \( J_2 \)-invariance constraints also have to be enforced:

\[
\delta a_{\text{inv}} = -\frac{2Dae}{\eta} \delta e_{\text{inv}}
\]

(3.29)

\[
\delta e_{\text{inv}} = \frac{(1 - e^2) \tan i}{4e} \delta i_{\text{inv}}
\]

(3.30)

Through algebraic manipulations, Eq. 3.25 to 3.30 can be combined to yield the set of orbit element corrections \( \Delta a_1, \Delta e_1 \) and \( \Delta i_1 \) that will lead to \( J_2 \)-invariant orbits at the first singularity:

\[
\Delta a_1 = \frac{-\eta \delta a_0 + 2Dae\delta e_0}{\eta - 2Dae (1 + e)}
\]

(3.31)

\[
\Delta e_1 = \frac{\eta (1 + e) \delta a_0 - 2Dae (1 + e) \delta e_0}{\eta \eta - 2Dae (1 + e)}
\]

(3.32)

\[
\Delta i_1 = \frac{4e (1 + e) \delta a_0 - 4ae \delta e_0}{\eta \eta \eta - 2Dae (1 + e) \tan i} - \delta i_0
\]

(3.33)
3.4. EXAMPLE OF THE $J_2$-INVARIANT RELATIVE ORBITS

The second condition requires that $\delta V_{a_2} = \delta V_{a_1}$, i.e. no tangential impulse at apoapsis is performed. This condition implies that:

$$\Delta v_{ta} = \frac{\eta a \eta}{4} \left( \frac{\Delta a}{a} - \frac{\Delta e}{1 - e} \right) = 0 \quad (3.34)$$

Combining Eq. 3.34 with Eq. 3.26 to 3.30 leads to a second set of corrections $\Delta a_2$, $\Delta e_2$ and $\Delta i_2$ that could be the most fuel-efficient way to reach $J_2$-invariance:

$$\Delta a_2 = \frac{-\eta \delta a_0 - 2Da \delta e_0}{\eta + 2De \left(1 - e\right)} \quad (3.35)$$
$$\Delta e_2 = \frac{-\eta \left(1 - e\right) \delta a_0 - 2Da \left(1 - e\right) \delta e_0}{\eta + 2De \left(1 - e\right)} \quad (3.36)$$
$$\Delta i_2 = \frac{-4e \left(1 - e\right) \delta a_0 + 4a \delta e_0}{\eta + 2De \left(1 - e\right)} \tan i - \delta i_0 \quad (3.37)$$

Finally, the third and last potential set of corrections is at $\delta V_{hi_2} = \delta V_{hi_1}$, i.e. no inclination correction is performed. Forcing $\Delta v_{hi_2} = 0$ imposes $\Delta i = 0$ and yields a third set of corrections to reach $J_2$-invariance conditions:

$$\Delta a_3 = -\delta a_0 - \frac{D \tan i}{2} \delta i_0 \quad (3.38)$$
$$\Delta e_3 = -\delta e_0 + \frac{\eta^2 \tan i}{4e} \delta i_0 \quad (3.39)$$
$$\Delta i_3 = 0 \quad (3.40)$$

Therefore, the most fuel-efficient way of reaching a $J_2$-invariant orbit is the $i^{th}$ set of conditions, out of the three, that will be the less expensive in terms of fuel. This fuel cost can easily be computed by mapping those corrections in the fuel equivalent-space and by computing the distance of the fuel-equivalent coordinates with respect to origin:

$$\delta V_i = S \begin{bmatrix} \Delta a_i \\ \Delta e_i \\ \Delta i_i \\ 0 \\ 0 \end{bmatrix} \quad (3.41)$$
with:
\[ d_{f_{ei}} = |\delta V_{r_{ei}}| + |\delta V_{\theta_{ei}}| + |\delta V_{\phi_{ei}}| \] (3.42)

Results can be further simplified by looking at the order of magnitude of the difference between the two first sets of corrections. The differences in terms of required corrections for the first two conditions are:

\[ \Delta a_1 - \Delta a_2 = \frac{-4\eta D e \delta a_0 - 8D^2 a e^2 \delta e_0}{\eta^2 - 4\eta D e^2 - 4D^2 e^2 + 4D^2 e^4} \] (3.43)

\[ \Delta e_1 - \Delta e_2 = \frac{2\eta^2 \delta a_0 - (8D^2 a e^2 + 4D^2 e^2 \eta - 8D^2 a e^4) \delta e_0}{a\eta^2 - 4a\eta D e^2 - 4D^2 a e^2 + 4a D^2 e^4} \] (3.44)

\[ \Delta i_1 - \Delta i_2 = \frac{(-16D e^2 - 8\eta^2 e + 16D e^4) \delta a_0 + 16a D e^2 \delta e_0}{a\eta \tan i (\eta^2 - 4\eta D e^2 - 4D^2 e^2 + 4D^2 e^4)} \] (3.45)

For typical 1 km size LEO formations, one can conservatively assume that:

\[ O(a) = 10^7 \text{ m} \] (3.46)
\[ O(e) = 10^{-2} \] (3.47)
\[ O(\delta a_0) = 10^2 \text{ m} \] (3.48)
\[ O(\delta e_0) = 10^{-3} \] (3.49)
\[ O(\delta i_0) = 10^{-3} \] (3.50)
\[ O(D) = 10^{-2} \] (3.51)

Under these assumptions, and assuming the orbit is not near-equatorial \((O(\tan i) \geq 1)\), the order of magnitude of the required corrections differences are:

\[ O(\Delta a_1 - \Delta a_2) = 10^{-2} \text{ m} \] (3.52)
\[ O(\Delta e_1 - \Delta e_2) = 10^{-5} \] (3.53)
\[ O(\Delta i_1 - \Delta i_2) = 10^{-8} \] (3.54)

At worst, these differences lead to an impact on the required fuel in the order of a few cm/s. Practically speaking, condition 1 and condition 2 lead to orbit element corrections that cannot be distinguished one from each other. Therefore, for inclined 1 km size LEO formations, the fuel cost needs to be computed for only two points. The most
3.4. EXAMPLE OF THE $J_2$-INARIANT RELATIVE ORBITS

TABLE 3.1 Chief Initial Orbit Elements

<table>
<thead>
<tr>
<th>$e_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
</tr>
<tr>
<td>$e_0$</td>
</tr>
<tr>
<td>$i_0$</td>
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<tr>
<td>$\Omega_0$</td>
</tr>
<tr>
<td>$\omega_0$</td>
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<tr>
<td>$M_0$</td>
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</tbody>
</table>

TABLE 3.2 Deputy Initial Orbit Elements Offset

<table>
<thead>
<tr>
<th>$\delta e_0$</th>
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<tbody>
<tr>
<td>$\delta a_0$</td>
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<tr>
<td>$\delta e_0$</td>
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<td>$\delta i_0$</td>
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<td>$\delta \Omega_0$</td>
</tr>
<tr>
<td>$\delta \omega_0$</td>
</tr>
<tr>
<td>$\delta M_0$</td>
</tr>
</tbody>
</table>

A fuel efficient way to reach a $J_2$-invariant orbit is the first set of corrections $\Delta a_1, \Delta e_1$ and $\Delta i_1$ (which is similar to $\Delta a_2, \Delta e_2$ and $\Delta i_2$) or the third set of corrections, $\Delta a_3, \Delta e_3$ and $\Delta i_3$, that requires no inclination corrections.

A numerical example is given next to illustrate the results. The reference orbit is described in Table 3.1. The deputy is given a small orbit element offset $\delta e_0$ as shown in Table 3.2. The problem is to find the closest $J_2$-invariant location starting from the initial location of the deputy.

The results of a systematic search is given in Fig. 3.1. The equations 3.31 to 3.33 and 3.38 to 3.40 avoid the need for a systematic search as they predict analytically the location of the singularities in the distance function where the minimum will be found. As expected, both condition 1 and condition 2 lead to practically the same correction. Both
singularities cannot be reasonably distinguished. They both are located at the “No tangential burns” singularity location. Therefore, the most fuel efficient way to reach a $J_2$-invariant relative orbit from the initial conditions of Table 3.1 and 3.2 is to perform the corrections $\Delta a = -0.38 \text{ m}$, $\Delta e = -5.8 \cdot 10^{-8}$ and $\Delta i = -8.0 \cdot 10^{-5}$ as given by equations 3.31 to 3.33.

3.5 Example of the Projected Circular Formation

The fuel-equivalent space theory can also be applied to geometrically defined formations such as the Projected Circular Formation (PCF). This theory can be applied to quickly identify the closest (in terms of fuel) position on a PCF without the need for a systematic search.

The PCF is a formation for which all members of the formation are at the same distance from the center of the formation in the normal-tangential plane (Fig. 3.2). In other words, as seen from Earth, all members are distributed on a circle. This could have several application for Earth observation.
Figure 3.2 Projected Circular Formation in Hill coordinates

In Hill coordinates, the projected circular formation is constrained by:

$$\rho^2 = \sqrt{y^2 + z^2} \quad (3.55)$$

where \(\rho\) is the radius of the PCF. All the admissible sets of Hill coordinates for a given \(\rho\) can be obtained by sweeping the circular formation angular position \(\beta\):

$$x(\beta) = -\frac{\rho}{2} \cos \beta \quad (3.56)$$

$$y(\beta) = \rho \sin \beta \quad (3.57)$$

$$z(\beta) = \rho \cos \beta \quad (3.58)$$

$$V_x(\beta) = \frac{\rho n}{2} \sin \beta \quad (3.59)$$

$$V_y(\beta) = \rho n \cos \beta \quad (3.60)$$

$$V_z(\beta) = \rho n \sin \beta \quad (3.61)$$

With an accurate and simple relative motion model, the required set of relative orbit elements \(\delta e_{pcf}(\beta)\) to reach the corresponding set of Hill coordinates at a desired orbit location can be obtained:

$$\delta e_{pcf}(\beta) = \Phi^{-1} \delta X(\beta) \quad (3.62)$$

where \(\delta X(\beta) = \begin{bmatrix} x & y & z & V_x & V_y & V_z \end{bmatrix}^T\). The matrix \(\Phi\) can be a simple linearization of the mapping between relative orbit elements and Hill coordinates, if the formation
is desired at the current location, or it can be a state transition matrix if the formation is required at another point farther on the orbit. The only requirement is that the matrix \( \Phi \) relates current relative orbit elements into Hill coordinates at the point where the projected circular formation is desired.

We seek the angle \( \beta_{\text{min}} \) for which the fuel cost of maneuvering from the current relative orbit elements \( \delta e_0 \) to \( \delta e_{\text{pcf}}(\beta) \) is minimized. Once the corresponding coordinates \( \delta V_1 \) and \( \delta V_{\text{pcf}}(\beta) \) have been identified in the fuel-equivalent space, the most fuel-efficient way to reach a projected circular formation starting from \( \delta e_0 \) is the set of elements \( \delta e_{\text{pcf}}(\beta) \) for which the distance between \( \delta V_1 \) and \( \delta V_{\text{pcf}}(\beta) \) is minimized.

The traditional way of solving this problem is to systematically compute the fuel cost to reach each location on the formation for the whole range of \( \beta \), i.e. between 0 and \( 2\pi \), such as done by Mueller [35]. However, the translation of the problem into the fuel-equivalent space leads to simple geometric relationships that can provide the closest location on a projected circular formation without the need for a systematic search. The problem is reduced to finding the minimal distance (and location of the minimal distance) between a point and an ellipse in a 6D space.

Theoretically, distances in the fuel-equivalent space are measured in terms of “fuel” distance \( d_{fe} \):

\[
d_{fe} = \sqrt{\delta V_p^2 + \delta V_{\rho}^2} + \sqrt{\delta V_a^2 + \delta V_{\alpha}^2} + \sqrt{\delta V_i^2 + \delta V_{\Omega}^2}
\]

(3.63)

instead of the more common Euclidean distance \( d_{Euc} \):

\[
d_{Euc} = \sqrt{\delta V_p^2 + \delta V_{\rho}^2 + \delta V_a^2 + \delta V_{\alpha}^2 + \delta V_i^2 + \delta V_{\Omega}^2}
\]

(3.64)

However, to be able to use the Euclidean distance provides the advantage that its derivative with respect to \( \beta \) is always continuous, which is not the case with the fuel distance.

Using the Euclidean distance systematically underestimates the fuel cost, as it assumes that all the impulses can be performed at the same time. However, there will always exist a constant \( K \) for which:

\[
d_{fe} = K d_{Euc}
\]

(3.65)
where

\[ 1 < K \leq \sqrt{3} \]  

(3.66)

\( K \) is equal to 1 if one of the three impulses \((\delta V_r, \delta V_a, \text{ or } \delta V_h)\) is infinitely larger than the other two. In this case, both distances computation become identical. On the other hand, \( K = \sqrt{3} \) if all three impulses are identical. It is unlikely to find both extremes in one single PCF. It is even more unlikely for this to happen for a small variation of \( \beta \). Therefore, it is reasonable to assume that \( K \) is approximately constant for a given PCF. This essentially means that both distance functions have the same “shape”, so that the \( \beta \) for minimum distance is located at the same place for the two functions. Therefore, finding the \( \beta \) for minimal \( d_{Euc} \) is a way of finding the \( \beta \) for minimum \( d_{fe} \).

If instead one seeks to minimize the Euclidean distance, this can be done by minimizing the function \( D \):

\[
D = \frac{1}{2} \delta V^T \delta V
\]  

(3.67)

where:

\[
\delta V = S [\delta e_0 - \Phi^{-1} \delta X (\beta)]
\]  

(3.68)

The distance \( D \) has the following first and second derivative expressions:

\[
\frac{dD}{d\beta} = \delta V^T S \Phi^{-1} \frac{d}{d\beta} \delta X (\beta)
\]  

(3.69)

\[
\frac{d^2 D}{d\beta^2} = (S\delta e_0)^T S \Phi^{-1} \delta X (\beta)
\]

\[
+ \left( S \Phi^{-1} \frac{d}{d\beta} \delta X (\beta) \right)^T \left( S \Phi^{-1} \frac{d}{d\beta} \delta X (\beta) \right)
\]

\[
- (S \Phi^{-1} \delta X (\beta))^T (S \Phi^{-1} \delta X (\beta))
\]  

(3.70)

The fully expanded \( D \) is of the form:

\[
D = A_0 + A_1 \cos(\beta) + B_1 \sin(\beta) + A_2 \cos(2\beta) + B_2 \sin(2\beta)
\]  

(3.71)

The coefficients of the distance function could be expressed in terms of \( \delta e_0, n, \rho \) and the components of \( \Phi^{-1} \). However the development of this expression would be tedious and the identification of the minimum of the function would not be straightforward.
Because of the sinusoidal nature of the function, and because first and second derivatives are known (and continuous), it is much more efficient to locate $\beta_{\text{min}}$ iteratively starting from an educated guess.

The location of the minimum of the function, where $dD/d\beta = 0$ and $d^2D/d\beta^2 > 0$ can be obtained very quickly with a Newton-Raphson iteration of the form:

$$\beta[k + 1] = -\frac{dD/d\beta}{d^2D/d\beta^2} + \beta[k]$$  \hspace{1cm} (3.72)

If the first guess is the $\beta$ which has the smallest distance out of a sufficiently large number (typically 6) of regularly-spaced initial guesses between 0 and $2\pi$, the algorithm usually converges to the function minimum within two or three iterations.

This is illustrated next by a numerical example. The chief orbit elements $e_0$ (the center of the formation) is set to a slightly elliptical 45 deg inclined low-Earth orbit, as described in Table 3.1. The deputy is given a small orbit element offset $\delta e_0$ as shown in Table 3.2. The problem consists in finding the angular position $\beta$ on the Projected Circular Formation that will be the less expensive to reach within a time frame of one orbit.

The "fuel" distance and the Euclidean distance for the whole range of $\beta$ is shown in Fig. 3.3. As can be seen on this figure, the use of the Euclidean norm systematically underestimates the fuel cost of the maneuver. However, both types of distance correctly locate the $\beta$ for minimum effort at $\beta = 1.01$. The "optimal" fuel cost of Figure 3.3 is the fuel cost of the maneuver, computed using optimal control theory [9] going from the initial position to the desired PCF location one orbit later at time $t_f$ and minimizing the cost function $J$:

$$J = \int_0^{t_f} \left( \frac{1}{2} u^T u \right) dt$$

where $u$ is the control vector. As opposed to the impulsive feedback controller theory, the optimal control theory assumes continuous firing of the thrusters and seeks to minimize a quadratic control effort. However, as shown in Fig. 3.3, both functions have their minimum located at the same $\beta$. This shows that the fuel-equivalent space theory
3.6 Conclusion

A new tool, the fuel-equivalent space, useful to rapidly compute the most fuel-efficient way to reach a desired formation has been presented. This theory is based on analytical and simple relations and is therefore well-suited for autonomous on-board application.

The method maps the required relative orbit elements corrections into a 6-dimensional fuel-equivalent space, in which a similar displacement on each of the axes requires the same amount of fuel. Therefore, the fuel cost of a given maneuver is minimized if the distance is minimized in the fuel-equivalent space.

Figure 3.3 Fuel Cost as a Function of PCF Angular Position

can be used even with continuous thrusters, assuming the optimal fuel cost difference between using continuous thruster firing and using impulsive thruster firing remains small for a given formation.

The same process can be applied to any type of formation that can be described geometrically. The use of geometric relations avoids the need for systematic search and transforms the problem into minimizing the distance between a point (the current location of the spacecraft) and a geometric shape (that represents the desired formation) at the cost of a few Newton-Raphson-type iterations.

3.6 Conclusion

A new tool, the fuel-equivalent space, useful to rapidly compute the most fuel-efficient way to reach a desired formation has been presented. This theory is based on analytical and simple relations and is therefore well-suited for autonomous on-board application.

The method maps the required relative orbit elements corrections into a 6-dimensional fuel-equivalent space, in which a similar displacement on each of the axes requires the same amount of fuel. Therefore, the fuel cost of a given maneuver is minimized if the distance is minimized in the fuel-equivalent space.
Two examples of application of the fuel-equivalent space theory have been presented. The first one is the \( J_2 \)-invariant relative orbits. In this case, finding the closest (in terms of fuel) \( J_2 \)-invariant relative orbit is reduced to computing the fuel cost of only two possible maneuvers, one of which is to perform no inclination maneuver. The second application is the Projected Circular Formation. Once mapped into the fuel-equivalent space, all the possible circular formation locations for a given formation radius form an ellipse in the 6-dimensional fuel-equivalent space. In this case, the problem is reduced to finding the minimum distance between a point and an ellipse. It has been shown that the Euclidean norm can effectively be used to locate the minimum of the fuel distance function. This distance has simple first and second derivatives and its global minimum can be identified with few iterations.

This theory can therefore be applied to any formation that can be geometrically defined in the relative orbit element space. Finding the most fuel-efficient way to reach the formation reduces to finding the minimum distance between a point and a geometric shape. If the formation has a geometrically simple shape (as is the case with the \( J_2 \)-invariant orbit and the PCF), simple analytical relations can be established and the most fuel-efficient maneuver can be identified analytically or with few simple iterations.
PART II

Autonomous Formation Flying Control
CHAPTER 4

Neighbouring Optimum Feedback Control Law
for Earth-Orbiting Formation Flying
Spacecraft


Submitted to *AIAA Journal of Guidance, Control, and Dynamics*.

Abstract

This paper presents the development of a neighbouring optimum feedback control law for formation flying spacecraft well suited for formation reconfiguration. The development of this controller makes use of the results of optimal control theory to obtain a formation flying control law that can perform a fuel/formation accuracy trade-off with the selection of only one gain. The controller is in the semi-analytic form, as only one time-varying gain matrix needs to be computed prior to the maneuver. Once this matrix is computed, the controller guarantees near-optimality for all the members of the formation. Simulation results compare the performance of this controller with other common formation flying control algorithms: the linear quadratic regulator and the mean orbit elements controller. Simulation results show that this neighbouring optimum controller can perform the same maneuver with a better accuracy while spending less propellant.
CHAPTER 4. NEIGHBOURING OPTIMUM FEEDBACK CONTROL LAW

4.1 Introduction

Formation flying of spacecraft has without any doubt gained a lot of interest within the engineering and scientific community in the recent years. This interest is most likely going to increase within the next years, mainly because of the numerous financial and operational advantages formation flying can procure.

For example, missions using formation flying spacecraft can potentially have a lower production cost due to economies of scale, in the case where a single large and complex satellite is replaced by several "mass production" smaller spacecraft. Secondly, using a constellation of spacecraft could decrease the cost of launch. Launching several smaller elements is potentially cheaper than launching a single big and heavy satellite, mainly because small satellites can be launched piggy-backed on a larger spacecraft flight support equipment.

Moreover, spacecraft formation flying presents several operational advantages. The most important one is an increased robustness through failure recovery and graceful degradation. In missions using multiple spacecraft in formation, if a subsystem failure occurs in one of the spacecraft, another fully functional spacecraft could support the disabled spacecraft. The capabilities can be shared. For example, when a power, communication or navigation system failure occurs in a spacecraft, it may be possible to use another spacecraft subsystem either by physically linking the spacecraft or by transmitting navigation information to the failed spacecraft. In the case of a separated spacecraft interferometer or a distributed antenna mission, the failure of one spacecraft would only cause a "graceful degradation" of the system, rather than compromising the whole mission. Thus, failure recovery and graceful degradation of the system decrease the risk of the mission. A second operational advantage is a mission restructuring capability. It is foreseeable to reconfigure the satellite formation on-orbit to follow new mission requirements. Moreover, if the mission has multiple objectives, resources can be optimized by dispatching a certain group of spacecraft having special
attributes to achieve one objective, and then command another group of spacecraft to achieve another objective in parallel.

However, using a formation of spacecraft involves several challenges. The first one is an increase in the required level of autonomy. In order to minimize the resources needed for ground support, it is required to limit the command inputs to the system to high-level commands to the whole formation. The formation would then have to autonomously define lower-level commands to each of the spacecraft. The second challenge is the design of a fuel-optimal control system. Obviously, the formation reconfiguration and maintenance control algorithms have to minimize the fuel consumption of every spacecraft of the formation. However, fuel consumption between the spacecraft also has to be balanced to maximize the lifetime of the complete formation, to ensure some spacecraft do no run out of fuel before the other ones.

The most studied formation flying control architecture is by far the "Leader/Follower" type of architecture. Under this architecture, the relative motion control problem is reduced to the tracking of a desired trajectory defined as a position relative to a reference trajectory. The guidance system is typically responsible for defining a reference trajectory (that could be based on the states of one member of the formation or any "virtual" point in space) and a position and velocity relative to this reference trajectory, and that, for all of the members of the formation. In this context, several types of controllers using continuous and variable thrust (as opposed to impulsive thrust) have been developed for different assumptions and different relative motion models.

The simplest, and most commonly used, relative motion model is the Clohessy-Wiltshire-Hill (CWH) model [46]. It is linear and models unperturbed relative motion about a circular reference orbit. By using the CWH model of relative motion, conventional linear control can be applied to Earth-orbiting formation flying. The main advantage of linear control theory is that it is a well-known method, with measurable performance and robustness assuming the linearization conditions are valid. For example, the Linear-Quadratic Regulator (LQR) uses a constant feedback gain matrix that minimizes the infinite-horizon state error and the quadratic actuator command. Ulybyshev
[52] evaluated the performance and robustness of a LQR to maintain a planar formation on a circular orbit. Other simulation results have also shown that the LQR can be applied to in-plane and out-of-plane maneuvers with reasonable fuel consumption even on an elliptical orbit. However, as is the case with many other systems, increasing the controller gains reduces the response time of the controller but with an increased fuel cost. This controller seems promising for long-term formation keeping that only implies small maneuvers and small deviations from the reference state.

Rahmani [38] also developed an optimal reconfiguration maneuver of two spacecraft assuming CWH dynamics. The main conclusion of the work is that a balanced fuel-optimal maneuver of two spacecraft on unperturbed circular orbit is achieved through equal and opposite acceleration of both spacecraft. However, those conclusions do not necessarily apply to elliptical and perturbed orbits. In fact, Inalhan [25] demonstrated that assuming that the reference orbit is circular, even when the eccentricity is as small as 0.005, leads to significant increase of fuel cost because the spacecraft "fights" the natural dynamics to keep the same relative trajectory as it would in a circular orbit.

On the other hand, the continuous mean orbit elements feedback control law, as developed by Schaub et al. [47, 46] controls the current mean orbit elements vector of the spacecraft toward the desired mean orbit elements vector and is well suited for elliptical reference orbits. With this controller, it is possible to "cooperate" with the physics of orbital dynamics. Acting directly on the mean orbit elements allows the control of specific orbit elements at specific instants on the orbit to increase the fuel efficiency of the algorithm. For example, it is much more fuel efficient to correct an inclination error at equator than at the pole, while an error in the ascending node is easier to compensate near the poles. By carefully choosing the time-varying gain matrix of the controller, those effects can be accounted for.

A continuous Cartesian coordinates feedback control law has also been proposed by Schaub and Junkins [46]. If the desired trajectory is described as an inertial position and an inertial velocity, a control feedback law based on Cartesian coordinates errors can be used. Assuming the relative orbits are $J_2$-invariant and that the distance be-
4.1. INTRODUCTION

tween the spacecraft is small, this simple feedback control law can make use of the non-linear dynamics (such as $J_2$-perturbed dynamics) to compensate position and velocity errors. A similar control law, but adaptive to slowly varying spacecraft masses, has also developed by de Queiroz et al [11].

The hybrid feedback law [46, 45] uses desired states defined as a set of orbit elements differences with a reference orbit, while the tracking errors are Cartesian coordinates errors. The main advantage of this method is that the controller uses inputs that are easily measured (relative position and velocity in orbital frame) while the reference is defined as orbit elements, which is more conveniently expressed than rapidly evolving Cartesian coordinates.

However, none of these feedback controllers can ensure optimality for elliptical reference orbits. The presence of several constraints in the problem, the non-linearity of the dynamics and the need for optimality makes the optimal control theory a candidate of choice for formation flying. This theory can fuel-optimize or time-optimize any reconfiguration maneuver while considering perturbed and non-linear dynamics. For circular reference orbits, an analytical solution can be obtained to get an analytical feedback law [33]. However, for reasonably complex dynamics, such as formation flying about an elliptical reference orbit, this method does require highly demanding numerical optimization iterations which could not be implemented on-board. However some near-optimal control methods, like the use of neighbouring optimal paths, only require to solve the optimal maneuver problem for one of the spacecraft. The other spacecraft of the formation can be considered as “neighbours” of this optimal path, and the resulting command offsets can be easily computed. Therefore, the complexity of the problem does not necessarily grow with the number of spacecraft in the formation, as is typically the case with optimal control solutions.

This paper therefore shows how a neighbouring optimal feedback controller can be applied to formation flying. This controller requires very few computation, which facilitates on-board implementation. Furthermore, the formation accuracy/fuel consumption trade-off can easily be implemented with the selection of only one weight
and the complexity of the control problem does not grow exponentially with the number of spacecraft in the formation. Moreover, as opposed to the other formation flying feedback controllers proposed until now, this controller ensures near-optimality for all the elements of the formation.

Section 4.2 first lays the theory upon which the neighbouring optimal controller is built. Then, section 4.3 applies the theory to the dynamics of formation flying. Finally, simulation results that assess the performance of the feedback controller in comparison with other common controllers and the optimal open-loop solution are presented in section 4.4.

### 4.2 Neighbouring Optimum Feedback Law Theory

The general theory behind the neighbouring optimum feedback control theory is summarized here for completeness. For a more detailed derivation of the theory, the reader is referred to the work of Bryson and Ho [9].

Optimal control theory provides a method for computing the control effort vector \( u(t) \) between the time \( t_0 \) and \( t_f \) that will minimize a cost function \( J \) of the form:

\[
J = \phi [\mathbf{x}(t_f)] + \int_{t_0}^{t_f} L [\mathbf{x}(t), u(t), t] \, dt
\]

(4.1)

where \( \mathbf{x}(t) \) are the states of the system and the dynamics of the system are governed by:

\[
\dot{\mathbf{x}}(t) = f [\mathbf{x}(t), u(t), t]
\]

(4.2)

The optimization of \( J \) can be done with or without a fixed terminal time \( t_f \) and with or without terminal constraints \( \psi [\mathbf{x}(t_f), t_f] = 0 \) on the terminal states. In this section, we shall consider that the terminal time \( t_f \) is fixed.

The system dynamics can be adjoined to the cost function with a vector of Lagrange multipliers \( \lambda(t) \) and \( \nu(t) \). For convenience, we define the Hamiltonian \( H \) as:

\[
H = L (\mathbf{x}, u, t) + \lambda^T f (\mathbf{x}, u, t)
\]

(4.3)
so that the system dynamics and the cost function are adjoined in an augmented cost function $\tilde{J}$:

$$\tilde{J} = [\phi + \nu^T \psi]_{t=t_f} + \int_{t_0}^{t_f} [L + \lambda^T (f - \dot{x})] \, dt \quad (4.4)$$

If one deliberately chooses the value of the Lagrange multipliers to be [9]:

$$\lambda^T = -H \dot{x} \quad (4.5)$$

where $H_x$ denotes the Jacobian of $H$ with respect to $x$, with the boundary conditions:

$$\lambda^T(t_f) = \phi_x(t_f) + \nu^T \psi_x(t_f) \quad (4.6)$$

then the variation in $\tilde{J}$ due to a variation of the control vector for fixed time $t_0$ and $t_f$ reduces to:

$$\delta \tilde{J} = \lambda^T(t_0) \delta x(t_0) + \int_{t_0}^{t_f} H_u \delta u \, dt \quad (4.7)$$

For an extremum, $\delta J$ must vanish for any arbitrary $\delta u(t)$, which leads to the condition

$$H_u = 0 \quad (4.8)$$

at any point in time between $t$ and $t_f$. Therefore, the optimal value of the control vector $u(t)$ and the resulting states $x(t)$ are obtained by solving the system of differential equations:

$$\dot{x} = f[x(t), u(t), t] \quad (4.9)$$

$$\dot{\lambda} = -f_u^T \lambda - L_x \quad (4.10)$$

with $u(t)$ determined by:

$$f_u^T \lambda + L_u = 0 \quad (4.11)$$

with the boundary conditions:

$$x(t_0) = x_0 \quad (4.12)$$

$$\lambda^T(t_f) = \phi_x(t_f) + \nu^T \psi_x(t_f) \quad (4.13)$$

and with the terminal conditions:

$$\psi [x(t_f), t_f] = 0 \quad (4.14)$$
Thus, solving this two-point boundary value problem yields a control sequence $u(t)$ that will bring the system from its initial states $x_0$ to its final location at time $t_f$ while minimizing the cost function $J$ and complying with the terminal constraints $\psi$. If an analytical solution to the problem can be found, then the control law $u(t)$ can become a feedback control law by replacing $t_0$ by $t$ and $x_0$ by $x(t)$ in the solution. However, for reasonably complex problems, such as formation flying dynamics about elliptical orbits, an analytical solution cannot be found. The system needs to be numerically solved at each time step, from $t$ to $t_f$ to obtain an optimal feedback control law. This is not suited for an on-board implementation. Therefore, alternative ways of simplifying the problem need to be found.

A less demanding solution (in terms of computational load) can be developed if the problem is linearized about the optimal path. The feedback control problem can be simplified if one instead seeks to minimize variations of the performance index about the optimal path. In this context, one has to minimize second-order variations of the cost function since all first-order terms necessarily vanish about the optimal path.

Perturbations from the optimal path produced by a small initial state offset $\delta x(t_0)$ and terminal conditions offset $\delta \psi$ are governed by the result of the linearization of the system (Eq. 4.9 to 4.14) around the optimal trajectory:

\begin{align*}
\delta \dot{x} &= f_x \delta x + f_u \delta u \\
\delta \dot{\lambda} &= -H_{x\lambda} \delta x - f_x^T \delta \lambda - H_{xu} \delta u \\
0 &= H_{u\lambda} \delta x + f_u^T \delta \lambda + H_{uu} \delta u \\
\delta \lambda(t_f) &= [(\phi_{xx} + (\nu^T \psi_x)_x) \delta x + \psi_x^T d\nu]_{t=t_f} \\
\delta \psi &= [\psi_x \delta x]_{t=t_f}
\end{align*}

where the notation $H_{xu}$ denotes the Jacobian of $H$ with respect to $x$ and $u$, such that:

\[
H_{xu} = \frac{\partial}{\partial u} (H_x)^T
\]
4.2. NEIGHBOURING OPTIMUM FEEDBACK LAW THEORY

In this context, we seek to minimize the second-order variation of the cost function \( J \), as the first-order variations already vanish about the extremal path:

\[
\delta^2 \bar{J} = \frac{1}{2} \left[ \delta x^T (\phi_{xx} + (\nu^T \psi_x)_x) \delta x \right]_{t=t_f} + \frac{1}{2} \int_{t_0}^{t_f} \left[ \delta x^T \delta u^T \right] \begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} dt
\]

Equations 4.15 to 4.19 form instead a linear two-point boundary value problem, which is much easier to solve analytically than the complete optimal control problem. Assuming \( H_{uu} \) is non-singular (which is always the case if \( H \) is appropriately defined), the problem can be restated as:

\[
\delta u(t) = -H_{uu}^{-1} (H_{uu} \delta x + f_u^T \delta \lambda)
\]

\[
\delta \dot{x} = A(t) \delta x - B(t) \delta \lambda
\]

\[
\delta \dot{\lambda} = -C(t) \delta x - A^T(t) \delta \lambda
\]

where:

\[
A = f_x - f_u H_{uu}^{-1} H_{ux}
\]

\[
B = f_u H_{uu}^{-1} f_u^T
\]

\[
C = H_{xx} - H_{ux} H_{uu}^{-1} H_{ux}
\]

This type of problem can be solved by the backward sweep method [9] and its solution takes the form:

\[
\delta \lambda(t) = S(t) \delta x(t) + R(t) d\nu
\]

\[
\delta \psi = R^T(t) \delta x(t) + Q(t) d\nu
\]

with the values of the \( S, R \) and \( Q \) matrices governed by:

\[
\dot{S} = -SA - A^T S + SBS - C
\]

\[
\dot{R} = -(A^T - SB) R
\]

\[
\dot{Q} = R^T BR
\]
and the boundary conditions:

\[
S(t_f) = \left[ \phi_x + (\nu^T \psi_x) \right]_{t=t_f} \\
R(t_f) = \left[ \psi_x \right]_{t=t_f} \\
Q(t_f) = 0
\] (4.34)

Through algebraic manipulations, Eq. 4.29 and 4.30 can be combined and substituted back into Eq. 4.23 to yield an expression of \( \delta u(t) \) in the form of a feedback law using the offset from the optimal trajectory \( \delta x \) and the desired terminal condition offset \( \delta \psi \) as inputs:

\[
\delta u(t) = -H_u^{-1} \left\{ [H_{ux} + f_u^T (S - RQ^{-1} R^T)] \delta x(t) + f_u^T RQ^{-1} \delta \psi \right\}
\] (4.37)

Thus, once \( S(t), R(t) \) and \( Q(t) \) are solved for the complete trajectory, Eq. 4.37 becomes a feedback control that leads to the reference terminal conditions or any small offset from these conditions (if desired) while minimizing the cost function \( J \).

### 4.3 Application to Formation Flight

The next step is to apply this theory to the formation flying control problem. Let:

\[
e = \left[ \frac{a}{R_e} \ e \ i \ \Omega \ \omega \ \nu \right]^T
\]

be the state vector, where \( a/R_e \) is the orbit semimajor axis normalized by Earth equatorial radius, \( e \) the orbit eccentricity, \( i \) the orbit inclination, \( \Omega \) the right ascension of the ascending node, \( \omega \) the argument of perigee and \( \nu \) the true anomaly. The system
dynamics $f$ are derived from Gauss Variational Equations:

$$f = \begin{bmatrix}
\frac{2a^2}{R_e h} \left( \varepsilon \sin(\nu) u_r + \frac{p}{r} u_t \right) \\
\frac{1}{h} \left\{ p \sin(\nu) u_r + \left[ (p + r) \cos(\nu) + re \right] u_t \right\} \\
\frac{r \cos(\theta)}{h} u_h \\
\frac{r \sin(\theta)}{h} u_h \\
\frac{1}{he} \left[ -p \cos(\nu) u_r + (p + r) \sin(\nu) u_t \right] - \frac{r \sin(\theta) \cos(\iota)}{h \sin(i)} u_h \\
\frac{h}{r^2} + \frac{1}{he} \left[ (p \cos(\nu) - 2re) u_r - (p + r) \sin(\nu) u_t \right]
\end{bmatrix} \tag{4.38}$$

where $h$ is the orbit angular momentum, $p$ the semi-latus rectum, $r$ the spacecraft distance from the Earth center, $\theta$ the argument of latitude and $u = [u_r \ u_t \ u_h]^T$ is the control vector with its elements being the control accelerations in the radial, transverse and normal directions respectively.

The cost function is designed to minimize both the quadratic control effort $u^T u$ and the quadratic relative orbit element error $(\Delta e)^T \Delta e$. Therefore:

$$L = \frac{1}{2} u^T \left( \text{diag} \left[ \rho_r \ \rho_t \ \rho_h \right] \right) u = \frac{1}{2} u^T \rho u \tag{4.39}$$

$$\phi(t_f) = \frac{1}{2} \Delta e^T \left( \text{diag} \left[ K_a \ K_e \ K_i \ K_{\Omega} \ K_\omega \ K_\nu \right] \right) \Delta e$$

$$= \frac{1}{2} \Delta e^T K \Delta e \tag{4.40}$$

It has been decided to implement a cost to a final state error (i.e. with $\phi \neq 0$ in Eq. 4.1) rather than imposing terminal conditions (i.e. $\psi(t_f) = 0$). This is to provide more freedom to the controller in the minimization of the cost. Indeed, the controller will thus be able to sacrifice the accuracy of the formation in order to minimize the propellant consumption, but always based on the relative values of $K$ and $\rho$. An infinite value of $K$ is theoretically identical to imposing a desired final state vector.

The only remaining missing information to synthesize the controller is the time evolution of the reference states and the Lagrange multipliers for the optimal (or nominal)
trajectory. The problem is largely simplified if one assumes that the trajectory of the "leader" (the reference trajectory), is uncontrolled and follows its natural motion. This does not mean that the elements of the formation have to remain uncontrolled, but rather means that all the elements of the formation evolve around an uncontrolled reference orbit, that could be either another spacecraft or a virtual point in space. This is the most probable scenario for a formation flying mission, as this should be the scenario that will most likely maximize the lifetime of the complete formation.

The dynamics of an uncontrolled spacecraft are simple to solve. If $\delta u = 0$, only the last element of $f$ (which represents $\dot{v}$) is not zero. This means that $\delta e(t)$ is constant except for the last element, which evolves with time assuming, of course, that the reference trajectory follows a Keplerian orbit. However, this model is only used to compute the gain matrices of the feedback control law, and the authors have noted that the use of a perturbed model in the computation of the gains (as opposed to a Keplerian orbit) has a negligible impact on the gains of the feedback control law and on its performance. For the sake of simplicity, only an unperturbed model of motion is used here for the computation of the controller gain $S$.

If the reference trajectory is uncontrolled, the dynamics of the states are decoupled from the Lagrange multipliers dynamics. Indeed, for an uncontrolled reference trajectory, $L_u = \rho u = 0$ so that enforcing Eq. 4.11 yields:

$$\lambda(t) = 0$$

which means that the Hamiltonian of the system is only a function of $u$. Consequently, all the derivatives of $H$ with respect to $x$ are 0:

$$H_{ux} = 0$$
$$H_{xu} = 0$$
$$H_{xx} = 0$$

and its second-order derivative is constant:

$$H_{uu} = \rho$$
Also, because $\psi(t_f) = 0$ and $\delta \psi = 0$:

\begin{align*}
R(t) &= 0 \\
Q(t) &= 0
\end{align*} 

(4.46)

(4.47)

which leaves us with only $S$ as unknown in Eq. 4.37.

From the dynamics of the system (Eq. 4.38), $f_u$ is:

\[
f_u = \begin{bmatrix}
\frac{2a^2 e \sin \nu}{h R_e} & \frac{2a^2 p}{h r R_e} & 0 \\
\frac{p \sin \nu}{h} & \frac{(p + r) \cos \nu + r e}{h} & 0 \\
0 & 0 & \frac{r \cos \theta}{h} \\
0 & 0 & \frac{r \sin \theta}{h \sin \iota} \\
\frac{-p \cos \nu}{h e} & \frac{(p + r) \sin \nu}{h e} & \frac{r \sin \theta \cos \iota}{h \sin \iota} \\
\frac{\eta (p \cos \nu - 2r e)}{h e} & \frac{\eta (p + r) \sin \nu}{h e} & 0
\end{bmatrix}
\] 

(4.48)

Moreover, because the reference trajectory is uncontrolled ($u = 0$), $f_x$ takes the simple form:

\[
f_x = -\frac{n (1 + e \cos \nu)}{\eta^3}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-3R_e (1 + e \cos \nu)}{2a} & e^2 \cos \nu + 3e + 2 \cos \nu & 0 & 0 & 0 & 2e \sin \nu
\end{bmatrix}
\] 

(4.49)

With all these simplifications, Eq. 4.31 becomes:

\[
\dot{S} = -S f_x - f_x^T S + (S f_u) H^{-1}_{uu} (f_u^T S)
\] 

(4.50)

with the final conditions:

\[
S(t_f) = K
\] 

(4.51)
From the form of Eq. 4.50 and 4.51, $S$ is obviously a $6 \times 6$ symmetric matrix if $K$ is symmetric (which is the case in the current application). This means that Eq. 4.50 is a system of 21 coupled differential equations, for which it seems unlikely to get an analytical solution. However, Eq. 4.50 can be integrated numerically backwards from $t_f$ to $t_0$ and the time evolution of $S(t)$ can be stored on-board since $S(t)$ only depends on the initial (or the final) location of the reference trajectory and the duration of the maneuver.

Once $S(t)$ is solved, Eq. 4.37 becomes a feedback law that can be applied on all the elements of the formation. Because the reference is uncontrolled, the command increment $\delta u(t)$ becomes the command $u(t)$ to apply to each element of the formation:

$$u(t) = -\rho^{-1} f_u^T S(t) \Delta e(t)$$  \hspace{1cm} (4.52)

where the orbit element error $\Delta e(t)$ is the difference between the current relative orbit element vector $\delta e(t)$ and the desired relative orbit elements $\delta e_d(t)$. This control law minimizes both the fuel consumption and the final orbit element error $\Delta e$ through the minimization of the cost function $J$. Thus, the selection of the values of $K$ and $\rho$ is a way of performing the trade-off between formation accuracy and propellant consumption.

Finally, the stability conditions of this controller are discussed. This can be done by looking at the conditions under which a Lyapunov function for the error dynamics exists [46]. Assuming the system is unperturbed, the error dynamics are:

$$\dot{\Delta e} = f_x \Delta e + f_u u$$  \hspace{1cm} (4.53)

Let $V$ be a Lyapunov function for the error dynamics:

$$V = \frac{1}{2} \Delta e^T \Delta e$$  \hspace{1cm} (4.54)

From Eq. 4.52 and 4.53, the Lyapunov function rate is:

$$\dot{V} = \Delta e^T \dot{\Delta e}$$  \hspace{1cm} (4.55)

$$\dot{V} = -\Delta e^T (\rho^{-1} f_u f_u^T S - f_x) \Delta e$$  \hspace{1cm} (4.56)
For the system to be stable, the function $\dot{V}$ needs to be negative semidefinite. Thus, the controlled system is stable if the matrix $(\rho^{-1} f_u f_u^T S - f_x)$ is positive semidefinite and asymptotically stable if the matrix $(\rho^{-1} f_u f_u^T S - f_x)$ is positive definite. The stability of the controller can therefore be assessed before the maneuver, since the values of $\rho$, $S$, $f_u$ and $f_x$ are known beforehand.

However, more simple stability conditions can be derived assuming the controller gains $S$ and $\rho^{-1}$ are sufficiently large. If this is the case, $f_x$ becomes negligible with respect to $\rho^{-1} f_u f_u^T S$. Hence, the stability condition is limited to $\rho^{-1} f_u f_u^T S$ being positive (semi-)definite. The matrix $\rho^{-1}$ is typically the identity matrix or the identity matrix multiplied by a positive gain. Therefore, without any loss of generality, we can redefine the condition for stability to $(f_u f_u^T) S$ being positive (semi-)definite. Because both $f_u f_u^T$ and $S$ are symmetric, $(f_u f_u^T) S = S (f_u f_u^T)$, which means that the product is positive (semi-)definite if and only if both $f_u f_u^T$ and $S$ are positive (semi-)definite. This leaves us mainly with the conclusion that in order for the controller to be globally stable, the gain matrix $S(t)$ must be positive definite for $t_0 < t < t_f$. Even though a rigorous demonstration that this is always the case cannot be provided, the gain matrix $S$ remained positive definite under all circumstances simulated by the authors. For typical LEO formations, the order of magnitude of the terms of $f_x$ and $f_u$ is $10^{-2}$ or smaller. This means that the order of magnitude of the product $f_u f_u^T$ is $10^{-4}$. Assuming $\rho$ is the identity matrix, the norm of the matrix $S$ should be of at least $10^4$ throughout the maneuver for this simplifying assumption to hold.

This algorithm is well-suited for reconfiguration maneuvers, as it will only associate an importance to the final relative orbit elements error. It can locally sacrifice formation accuracy to save propellant for moments of the maneuver where propellant will be efficient, as it is aware of the dynamics of the system through $f_u$ and $S$. Other types of controllers such as the LQR or the mean orbit element controller only have a limited knowledge of the dynamics of the system and have no clue about the duration of the maneuver. They therefore cannot guarantee optimality (or sub-optimality) of the reconfiguration.
However, the neighbouring optimum feedback law developed here is not fully analytical. An analytical version of this feedback controller would require an analytical solution to $S(t)$, which seems very difficult to obtain, if at all possible. Since its computation assumes unperturbed motion of the reference trajectory, it is only a function of the controller weights $K$ and $\rho$, the initial position of the reference at the beginning of the trajectory and the duration of the maneuver. Therefore, whatever is the number of spacecraft in the formation, $S(t)$ only needs to be solved once, and the controller will guarantee a near-optimal result for all the members of the formation, as long as they remain sufficiently close from the uncontrolled reference trajectory.

### 4.4 Simulation Results

In this section, we shall compare the neighbouring optimal feedback law with other common formation flying control algorithms, namely a traditional LQR and the mean orbit elements controller. Results will be compared for a typical formation flying problem: the reconfiguration from an arbitrary location into a Projected-Circular Formation (PCF) performed over a duration of one orbit.

The reference, or "chief", initial orbit elements $e_0$ are given in Table 4.1. This reference orbit starts from the initial states of Table 4.1 and follows a natural uncontrolled motion until the end of the maneuver one orbit later. The "deputy" spacecraft that performs the maneuver starts with the relative state vector $\delta e_0$, as described in Table 4.2. The relative orbit elements of the deputy $\delta e$ is the difference between the orbit elements of the deputy $e_d$ and the orbit elements of the chief $e_c$. It shall be noted that in this section, relative motion is described in terms of relative mean anomaly $\delta M$, instead of relative true anomaly $\delta \nu$. This is common in the formation flying literature since relative mean anomaly remains naturally constant for unperturbed and uncontrolled orbits with the same orbital energy. However, the neighbouring optimum feedback controller applied later in this section still expects a true anomaly error as input, as the dynamic model used in the computation of the controller gains uses $\nu$ as the 6th state variable.
### TABLE 4.1 Chief Initial Orbit Elements

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<thead>
<tr>
<th>$e_0$</th>
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<tbody>
<tr>
<td>$a_0$</td>
<td>1.1$R_e$</td>
</tr>
<tr>
<td>$e_0$</td>
<td>0.05</td>
</tr>
<tr>
<td>$i_0$</td>
<td>$\pi/4$</td>
</tr>
<tr>
<td>$\Omega_0$</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_0$</td>
<td>0</td>
</tr>
<tr>
<td>$M_0$</td>
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</tbody>
</table>

### TABLE 4.2 Deputy Initial Orbit Elements Offset

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</thead>
<tbody>
<tr>
<td>$\delta a_0$</td>
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</tr>
<tr>
<td>$\delta e_0$</td>
<td>+0.0001</td>
</tr>
<tr>
<td>$\delta i_0$</td>
<td>+0.0001</td>
</tr>
<tr>
<td>$\delta \Omega_0$</td>
<td>-0.0001</td>
</tr>
<tr>
<td>$\delta \omega_0$</td>
<td>-0.0001</td>
</tr>
<tr>
<td>$\delta M_0$</td>
<td>+0.0001</td>
</tr>
</tbody>
</table>
TABLE 4.3 Chief Final Orbit Elements

<table>
<thead>
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<th>$e_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_f$</td>
</tr>
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<td>$e_f$</td>
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<tr>
<td>$i_f$</td>
</tr>
<tr>
<td>$\Omega_f$</td>
</tr>
<tr>
<td>$\omega_f$</td>
</tr>
<tr>
<td>$M_f$</td>
</tr>
</tbody>
</table>

TABLE 4.4 Deputy Desired Orbit Elements Offset after 1 Orbit

<table>
<thead>
<tr>
<th>$\delta e_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta a_f$</td>
</tr>
<tr>
<td>$\delta e_f$</td>
</tr>
<tr>
<td>$\delta i_f$</td>
</tr>
<tr>
<td>$\delta \Omega_f$</td>
</tr>
<tr>
<td>$\delta \omega_f$</td>
</tr>
<tr>
<td>$\delta M_f$</td>
</tr>
</tbody>
</table>

The final reference orbit elements, after one orbit, are given in Table 4.3. These elements are the result of a one-orbit propagation considering the effect of the $J_2$ perturbation. The final desired relative orbit elements of the deputy are given in Table 4.4. These relative states would locally place the deputy on a 1 km PCF around the reference trajectory. The control problem therefore consists of maneuvering the deputy from $\delta e_0$ to $\delta e_f$ at $t_f = 5840$ s, i.e. one orbit later.

The first controller with which the neighbouring optimum feedback law is compared is the conventional LQR controller. This controller is designed to use the Hill coordinates (relative position and velocity in the orbital frame) as inputs and outputs a control acceleration in the orbital frame. It is synthesized using the linearized CWH model of
relative motion, which can be written as:

\[
\begin{bmatrix}
\dot{X} \\
\dot{V}
\end{bmatrix} = 
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
3n^2 & 0 & 0 & 0 & 2n & 0 \\
0 & 0 & 0 & -2n & 0 & 0 \\
0 & 0 & -n^2 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
X \\
V
\end{bmatrix}
\] (4.57)

where \( n \) is the orbital mean motion of the chief and \( X \) and \( V \) are respectively the relative position and velocity in the radial, transverse and normal directions. Obviously, the CWH model assumes a circular reference orbit which will inevitably affect the performance of the controller in this relatively large-eccentricity simulation scenario. Nevertheless, the LQR represents one of the simplest controllers that can be implemented in this context and can be used as a reference. This infinite-horizon type of controller seeks to minimize the cost function

\[
J = \int_0^\infty \left[ (\Delta x)^T Q (\Delta x) + u^T R u \right] dt
\] (4.58)

where \( \Delta x \) is the position and velocity error vector and \( Q \) and \( R \) are weighting matrices. In this particular scenario, \( Q \) was set as the identity matrix and \( R \) is a diagonal matrix with \( 10^9 \) as non-zero elements. The result is a static gain matrix \( K \) that computes the control commands from the Hill coordinates errors:

\[
u = -K \Delta x
\] (4.59)

The second benchmark controller is the mean orbit elements controller [47, 46]. As opposed to the LQR, the mean orbit elements controller uses orbit-element errors as input. The control of orbit elements instead of Hill coordinates provides the main advantage of having a controller that uses more wisely the laws of orbital dynamics. By having some gains high or low at certain points of the orbit, it is possible to correct some orbit element errors at a location where the control command is efficient. In one of its simplest forms, the mean orbit elements controller can be expressed as:

\[
u = -P \left( f_u^T f_u \right)^{-1} f_u \Delta e
\] (4.60)
The matrix $P$ is a time-varying gain matrix. The only constraint in the demonstration of the stability of this controller is for $P$ to be positive definite and sufficiently large [46]. We choose here to define $P$ as:

$$
P = \begin{bmatrix}
P_{a_0} + P_{a_1} \cos^N f \\
P_{e_0} + P_{e_1} \cos^N f \\
P_{i_0} + P_{i_1} \cos^N \theta \\
P_{\Omega_0} + P_{\Omega_1} \sin^N \theta \\
P_{\omega_0} + P_{\omega_1} \sin^N f \\
P_{M_0} + P_{M_1} \sin^N f
\end{bmatrix}
$$

(4.61)

The values of the coefficients of the $P$ matrix were heuristically set to $P_{a_0} = P_{e_0} = P_{i_0} = P_{\Omega_0} = P_{\omega_0} = P_{M_0} = 0.001$, $P_{a_1} = P_{e_1} = P_{i_1} = P_{\Omega_1} = P_{\omega_1} = P_{M_1} = 0.001$ and $N = 6$, as it seemed the best trade-off to minimize both fuel and state error over a 1 orbit period. The selection of the optimal value of these 13 variables is within itself a very challenging task [47] and it is not the purpose of this study to find neither the optimal values nor a way to obtain them. However, the performance of the controller with this set of gains seemed to be typical of the type of performance that can be expected from this controller.

Finally, for the neighbouring optimum feedback law, $S(t)$ was computed off-line and then stored. Once the trajectory of the reference is predicted, Eq. 4.50 is solved using Eq. 4.51 as boundary conditions. The values of the weighting matrices, which are the only values to tune in the controller were set to:

$$
K = 10^6 \cdot I
$$

(4.62)

$$
\rho = I
$$

(4.63)

In simulation, the value for $S$ at each time step is interpolated from a look-up table. A block-scheme representation of the implementation of the controller is given in Fig. 4.1. As shown in this figure, the only inputs required are the current orbit element vector of the deputy $e_d$, the desired set of elements $e_{des}$ and the current maneuver time $t$, required to interpolate the value of $S(t)$. Formally speaking, the chief orbit elements $e_c$
should be used to compute $f_u^T$ instead of the deputy state vector in Eq. 4.52. However, because the absolute orbit elements of the chief and the deputy are so close, the use of one or the other in the computation of $f_u^T$ has little impact on the performance of the controller. The matrix $f_u(e_c)$ is essentially identical to $f_u(e_d)$. To use $e_d$ instead reduces the number of inputs of the controller.

For all three control algorithms, the desired trajectory $\delta e_{des}(t)$, from which the error is computed, is the trajectory that would "naturally" (i.e. without any control effort) lead to the desired position at time $t_f$. This trajectory can be computed with any state-transition matrix that predicts relative motion in the form:

$$\delta e_{des}(t) = \Phi^{-1}(t)\delta e_f$$

(4.64)

where $\Phi(t)$ is a matrix that maps the current relative orbit elements to the set of relative orbit elements at time $t_f$. In simulation, a linearized model for $J_2$-perturbed elliptical orbits was used [19, 20]. Therefore, once the error between $\delta e$ and $\delta e_{des}$ is reduced to zero, no further control effort is required to bring the spacecraft to its desired location at $t = t_f$. 

**Figure 4.1** Block-Scheme of the Neighbouring Optimum Feedback Law Implementation
The resulting control command history for each of the controllers are given in Fig. 4.2 to 4.4 (each figure uses a different scale to highlight the command history profile). The time history of the quadratic error ($\Delta e^T \Delta e$ at $t = t_f$), between the desired “natural” trajectory and the actual trajectory for each of the controller is shown at Fig. 4.5. A comparison of the fuel cost (cumulated required velocity impulse), the quadratic effort $U$:

$$U = \int_{t_0}^{t_f} (u^T u) \, dt$$  \hspace{1cm} (4.65)

and the final error for each of the algorithms is presented in Table 4.5. The algorithms are also compared with the optimal control theory open-loop solution [9] when imposing the final desired orbit elements as constraints to the problem and minimizing the quadratic control effort $U$. This type of controller is not a feedback controller. However, it is used here as a baseline since it provides the optimal solution, assuming both modelled dynamics and true dynamics are similar.

As can be seen in Table 4.5, the neighbouring optimum feedback law provides a better accuracy (in terms of final quadratic error) than the two other controllers with a smaller fuel cost (both in terms of total absolute velocity impulse and total quadratic effort). The reader should note however that this is not a control algorithm compari-
4.4. SIMULATION RESULTS

Figure 4.3 Mean Orbit Elements Controller Command Signal History

Figure 4.4 Neighbouring Optimum Feedback Controller Command History
Figure 4.5 Orbit Elements Quadratic Error Time History

<table>
<thead>
<tr>
<th></th>
<th>LQR</th>
<th>Mean Orbit Elements Controller</th>
<th>Neighbouring Optimum Feedback Controller</th>
<th>Open-loop Optimal Command</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fuel Cost (m/s)</td>
<td>3.9324</td>
<td>0.8181</td>
<td>0.4770</td>
<td>0.4721</td>
</tr>
<tr>
<td>Quadratic Control Effort (m²/s³)</td>
<td>0.0186</td>
<td>1.6363×10⁻⁴</td>
<td>4.1250×10⁻⁵</td>
<td>4.0851×10⁻⁵</td>
</tr>
<tr>
<td>Quadratic Final Error</td>
<td>4.5376×10⁻¹⁰</td>
<td>1.7918×10⁻⁹</td>
<td>2.0837×10⁻¹²</td>
<td>0</td>
</tr>
</tbody>
</table>
son campaign. It could be possible to obtain slightly better performance with the other two algorithms with more finely tuned gains. However, it is the authors' opinion that the simulation results presented here are typical of what can be expected from the controllers. Therefore, these results are used as an indicator of relative performance.

By comparing Fig. 4.2, 4.3 and 4.4, one can note that the neighbouring optimum feedback law has a relatively constant but small command magnitude, while the two other feedback controllers have higher command acceleration magnitudes. This is especially true for the LQR that will try to correct the error right from the start and will maintain a relatively large steady-state error (approx. $10^{-8}$) for the remaining minutes of the maneuver. This steady-state error is inevitable because of the elliptical nature of the reference orbit (the LQR design assumed a circular reference orbit) and the presence of perturbations caused by $J_2$ harmonics. The neighbouring optimum feedback law is the only controller, out of the three, that is able to sacrifice formation accuracy at the beginning of the trajectory to get pay-offs in terms of propellant and accuracy at the moment where the formation is needed.

It is also interesting to observe that the neighbouring optimal feedback law basically recreates the optimal open-loop command, which demonstrates once again the near-optimality of the neighbouring optimum feedback law. As can be seen in Fig. 4.6, the time-history of the optimal open-loop command is similar to the output computed by the feedback law, especially during the first part of the trajectory. This explains why the performance in terms of fuel consumption of the neighbouring optimum feedback law is so close to the performance of the open-loop optimal command computed off-line.

### 4.5 Conclusion

This paper showed how results of the optimal control theory can be applied to formation flight to get a neighbouring optimum feedback law. The controller is of the semi-analytical form, as only one time-varying gain matrix needs to be numerically computed once. However, once this matrix is known, the controller provides essen-
Eventually optimal results for all the elements of the formation, assuming they orbit sufficiently close from the uncontrolled reference trajectory. This type of controller is especially well-suited for on-board implementation where "snap-shot" types of formation are needed, i.e. when a specific formation is needed at a specific point of the orbit, disregarding the formation accuracy during the maneuver.

The performance of this controller has been compared with two other common formation flying controllers, namely the LQR and the mean orbit elements controller in a typical PCF reconfiguration problem. The proposed neighbouring optimum feedback law yielded a better accuracy while maintaining a smaller fuel consumption, both in terms of absolute velocity impulse and quadratic effort.

Future work on the controller would be to identify solution paths to obtain an analytical solution to the $S$ matrix or under which assumptions such a solution can be obtained. This would lead to a fully analytical feedback control law that guarantees near-optimality for all the members of the formation. Furthermore, collision avoidance requirements could be added as an additional constraint to the optimization problem.
PART III

Autonomous Guidance and Control Loop
CHAPTER 5

Autonomous Guidance & Control of Earth-Orbiting Formation Flying Spacecraft: Closing the Loop


Abstract

Previous work on autonomous formation flying guidance and control identified three key challenges to overcome in order to obtain a fully autonomous guidance and control loop: an accurate but simple model of relative motion about elliptical and perturbed orbits, an efficient way of performing conflicting requirements trade-off with power-limited on-board computers, and finally an optimal or near-optimal control algorithm easy to implement on a flight computer. This paper first summarizes recent developments on each of these subject that help to overcome these challenges, developments which are then used as building blocks for an autonomous formation flying guidance and control system. This system autonomously performs trade-offs between conflicting requirements, i.e. minimization of fuel cost, formation accuracy and equal repartition of the fuel expenditure within the formation. Simulation results show that a complete guidance and control loop can be established using mainly analytical results and with very few numerical optimization which facilitates on-board implementation.
5.1 Introduction

Formation flying will be a major trend in the upcoming years in space exploration. Indeed, replacing “conventional” large and expensive monolithic spacecraft by several smaller and cheaper spacecraft presents several advantages.

Firstly, formation flying presents operational advantages, such as mission reconfiguration capability and robustness to system failures through failure recovery and graceful degradation. In deep-space missions using multiple spacecraft in formation, if a sub-system failure occurs in a spacecraft, another fully functional spacecraft could support the disabled spacecraft. The capabilities can be shared. For example, when a power, communication or navigation system failure occurs in a spacecraft, it may be possible to use another spacecraft sub-system either by physically linking the spacecraft or by transmitting navigation information to the failed spacecraft. In the case of a distributed spacecraft interferometer or a distributed antenna mission, the failure of one spacecraft would only cause a “graceful degradation” of the system, rather than compromising the whole mission. The second operational advantage is a mission restructuring capability. It is foreseeable to reconfigure the satellite formation on-orbit to follow new mission requirements. Moreover, if the mission has multiple objectives, resources can be optimized by dispatching a certain group of spacecraft having special attributes to achieve one objective, and then command another group of spacecraft to achieve another objective in parallel.

Moreover, formation flying presents financial advantages. Such a mission can potentially have a lower production cost due to economics of scale, in the case where a single large and complex satellite is replaced by several “mass production” smaller spacecraft. Secondly, using a constellation of spacecraft could decrease the cost of launch. Launching several smaller elements is potentially cheaper than launching a single big and heavy satellite, mainly because small satellites can be launched piggy-backed on a larger spacecraft flight support equipment.
However, using a formation of spacecraft involves several challenges. The main one is an increase of the required level of autonomy. In order to minimize the resources needed for ground support, it is required to limit the command inputs to the system to high-level commands to the whole swarm of spacecraft. The swarm of spacecraft would then have to autonomously define lower-level commands for each of the spacecraft. However, these low-level commands would have to remain fuel-optimal to maximize the lifetime of the whole formation. Moreover, if the consumption of fuel is not well balanced between spacecraft, some spacecraft could run out of fuel before other ones, and cause a premature degradation of the performance of the system.

This paper follows a paper presented in 2006 [16] where the main challenges in achieving a completely autonomous formation flying guidance and control system were presented. In order to be autonomous and implemented on-board, such guidance and control algorithms have to require as few computation as possible (to facilitate implementation on limited-capacity on-board computers) and require no (or few) inputs from the ground segment. This paper summarizes recent developments which solve some of the problems highlighted in the first paper and describes a new way of using these tools in a completely autonomous guidance and control loop.

Section 5.2 summarizes recent developments in relative motion theory that lead to an analytical model of relative motion for $J_2$-perturbed elliptical orbits. Then, section 5.3 presents the fuel-equivalent space, a tool used to rapidly and analytically predict the fuel cost of maneuvers and/or rapidly identify the most fuel-efficient way to attain a given formation. Section 5.4 summarizes the results of the neighbouring-optimum controller, a method for autonomous formation flying control. Section 5.5 describes how all these tools can be stitched together in a formation flying guidance and control system that autonomously performs trade-offs between the formation flying conflicting requirements using a typical projected-circular formation as an example. Finally, section 5.6 shows simulation results and demonstrates how a trade-off between conflicting requirements can be done in a typical formation flying reconfiguration problem.
5.2 Relative Motion Theories

Autonomous guidance and control systems require accurate but simple models of reality in their algorithms. Models have to be accurate enough to prevent unnecessary fuel expenditure, but simple enough to allow on-board implementation. If perturbation models are included in the on-board model of reality, natural motion induced by the perturbations can be used to support maneuvers. If these perturbations are not included, the guidance and control system will most likely compensate for these perturbations, therefore leading to an unnecessary fuel expenditure.

The most widely used relative orbital motion model is the Clohessy-Wiltshire-Hill model [54, 40, 30]. It provides a time-explicit closed-form analytical solution to relative motion problem for circular unperturbed orbits. This model provides significant insights into the natural relative motion about circular and unperturbed reference orbits. However, assuming a circular reference orbit yields considerable errors when the eccentricity of the reference orbit grows [25]. Several models have therefore been proposed to model relative motion about unperturbed elliptical orbits [42, 7, 25, 58]. In a recent publication, Lane and Axelrad [27] developed a time-explicit closed-form solution and studied the relative motion for bounded relative elliptical orbits. Melton [32] also proposed an alternative solution for small-eccentricity orbits.

However, these models do not take into account orbit perturbations. The most important perturbation encountered for the relative motion problem, and also the most studied, is the perturbation caused by the oblateness of the Earth, referred to as the $J_2$ perturbation. Schweighart and Sedwick [48, 49] modified the classic Clohessy-Wiltshire-Hill model to include the orbit-averaged impact of the $J_2$ perturbation on a circular reference orbit. The most challenging problem is nevertheless to consider an elliptical and perturbed reference orbit. The most accurate way to model this problem is of course with numerical models [26, 4]. In this case, solutions to the relative motion problem are obtained through numerical integration of the dynamics equations. However, numerical methods are not well suited for autonomous on-board applications because
they typically require large computing effort. Few publications actually provide an analytical solution to the relative motion around elliptical reference orbits taking into account the $J_2$ perturbation.

Gim and Alfriend [13] solve the problem by proposing a state transition matrix that provides a time explicit solution for the relative motion about a $J_2$-perturbed elliptical orbit. This model provides an accurate solution to the problem. However, even though the model is fully analytical, the elements of the state transition matrices remain quite complex, the states of the reference trajectory still need to be numerically computed and matrix products and inversions remain. On the other hand, Schaub studied the relative motion about elliptical reference orbits under $J_2$ perturbation with very simple expressions using classical orbit elements [42]. However, this analysis is only performed in the mean orbit elements space. This model cannot be written readily into a state transition matrix form as the mapping between instantaneous, or osculating, orbit elements remains to be done.

Some recent work by the authors led to an analytical state transition matrix that accurately models relative motion about elliptical reference orbits under $J_2$ perturbation, while using simpler expressions and without the need to numerically propagate the states of the reference trajectory [19, 20]. It builds upon the approach of Schaub [42], but bridges the gap between osculating relative motion and relative mean orbit element drift. Desired formation relative dynamics are typically commanded in terms of osculating, or “actual”, relative dynamics, which is why it is relevant to describe the relative motion in terms of osculating elements instead of mean elements. This simplified model is oriented toward an on-board implementation for mission scenarios where computational power is limited, such as low-cost and low-power scientific missions.

The proposed model uses a geometric approach, similar to the work of Gim-Alfriend [13] but with some simplifying assumptions. The model neglects variations in the short-periodic relative motion induced by the $J_2$ perturbation between the deputy and the chief, but includes a osculating to mean orbit elements mapping. In other words, it “adds” the relative mean orbit element drift to the natural osculating-element Keple-
rian dynamics, neglecting the impact of short-period variations on the relative motion. This simplification is made at the cost of a prediction error as large as the short-periodic terms variations between the deputy and the chief. For two spacecraft orbiting very close from one another, this error will remain small as the short-period oscillations caused by the $J_2$ perturbation will be nearly the same for both spacecraft. However, in all cases, this error will remain bounded even for long-term prediction.

The main advantage of this approach is that the states of the reference trajectory at the true anomaly where the relative dynamics need to be known are not required. All the elements of the state transition matrix are computed from the initial position of the reference trajectory and the true anomaly for which the relative motion needs to be predicted. Models that take into account short-period variations [13] need the states of the reference at the final time to do an accurate mapping between the mean elements and the osculating elements at this location.

The main use of such a model in the guidance and control loop is to predict which relative state vector $\delta e_0$ is required to reach without any control effort the relative Hill coordinates $\delta X$ at a given true anomaly $\nu$, given the state vector of the reference trajectory (sometimes referred to as “leader”):

$$\delta e_0 = \Phi(e_0, \nu)^{-1}\delta X(\nu)$$

(5.1)

where the elements of the state transition matrix $\Phi$ assume a $J_2$-perturbed elliptical reference orbit [19, 20]. For circular reference orbits, other models are better suited [48, 49].

This state transition matrix is the first building block of the autonomous guidance and control system presented here, as it analytically provides which relative orbit elements are required to “naturally” reach a desired set of Hill coordinates at any given true anomaly.
5.3 Fuel-Equivalent Space

Another recently-developed tool, the fuel-equivalent space [18, 22], is the second building block of the guidance and control loop. The fuel-equivalent space is a mathematical space where displacements on every axis are identical in terms of fuel cost. Thus, minimizing the fuel cost of a maneuver is equivalent to minimizing the distance in the fuel-equivalent space. Therefore, minimum-fuel problems are reduced to simple geometric problems in the fuel-equivalent space.

This theory builds upon the results of the impulsive feedback controller [43, 46] which was proposed as a way to perform orbit element corrections while minimizing the impact on the other orbit elements. It makes use of the Gauss variational equations and can perform any arbitrary small orbit correction with only three impulses. If only one or two elements are to be corrected, the controller provides essentially optimal results in terms of fuel. If all six elements are to be corrected, the controller proposes maneuvers that are only a few percents larger than the optimal multi-impulse solution. However, the most important advantage of this technique is that the impulses and their location can be computed analytically with simple expressions, leading quickly to a good approximation of the fuel-cost of a maneuver, even if the spacecraft does not make use of impulsive thrusters.

The fuel-equivalent space maps the relative orbit element state vector $\delta e$ into six fuel-equivalent coordinates

$$
\delta V = \begin{bmatrix}
\delta V_r \\
\delta V_o \\
\delta V_h \\
\delta V_{ho} \\
\delta V_p \\
\delta V_o
\end{bmatrix}
$$

(5.2)

where the coordinates represent the components of the three velocity impulses suggested by the impulsive feedback controller. The fuel-equivalent coordinates are computed through the linear mapping:

$$
\delta V = S \delta e
$$

(5.3)

where the elements of $S$ are defined by the state vector of the reference trajectory. As presented in Ref. [18], the distance between $\delta V_1$ and $\delta V_2$ such that $\delta V_1 = S \delta e_1$ and
\[ \delta V_2 = S \delta e_2 \] is a good approximation of the fuel cost, in terms of velocity impulse, required to go from \( \delta e_1 \) to \( \delta e_2 \). However, the distance in the fuel-equivalent space is not the conventional Euclidean distance. In fact, because fuel-equivalent coordinates relate to the magnitude of the impulses predicted by the impulsive feedback controller, and because these impulses occur at different locations of the orbit, simultaneous displacements in some of the planes of the fuel-equivalent space are not possible. This so-called "fuel-equivalent" distance \( d_{fe} \) between \( \delta V_1 \) and \( \delta V_2 \) takes the form:

\[
d_{fe} = \sqrt{(\delta V_{r_{a2}} - \delta V_{r_{a1}})^2 + (\delta V_{t_{a2}} - \delta V_{t_{a1}})^2}
+ \sqrt{(\delta V_{r_{p2}} - \delta V_{r_{p1}})^2 + (\delta V_{t_{p2}} - \delta V_{t_{p1}})^2}
+ \sqrt{(\delta V_{h_{z2}} - \delta V_{h_{z1}})^2 + (\delta V_{h_{\Omega2}} - \delta V_{h_{\Omega1}})^2}
\]  

(5.4)

This mapping is thus a very efficient way to estimate the most fuel-efficient way to reach a formation because the problem is reduced to finding the minimum distance between a point (current spacecraft location) and a geometrical shape (the desired formation). For example, in the fuel-equivalent space, all \( J_2 \)-invariant relative orbits \([42,46]\) form a straight line, while all the relative states forming a projected-circular formation about a leader form a six-dimension ellipse. It will be shown later that this tool can be used in an autonomous formation flying guidance and control system, as it is a straightforward way of predicting the fuel cost of a maneuver.

### 5.4 Relative Motion Control

The last missing element for a completely autonomous guidance and control loop was a simple but fuel-efficient control algorithm. Such an algorithm will have to efficiently and accurately bring the elements of the formation from their initial set of relative orbit elements to the desired relative orbit elements within a reasonable time frame with as few computation as possible.

Most of the formation flying control algorithms found in the literature assume the leader/follower type of architecture. Under this architecture, the relative motion control
problem is reduced to the tracking of a desired trajectory defined as a position relative to a reference trajectory. The guidance system is typically responsible for defining a reference trajectory (that could be based on the states of one member of the formation or any "virtual" point in space) and a position and velocity relative to this reference trajectory.

The simplest controllers assume unperturbed relative motion about a circular reference orbit. By using the CWH linear relative motion model, conventional linear control can be applied to Earth-orbiting formation flying spacecraft. The main advantage of linear control theory is that it is a well-known method, with measurable performance and robustness assuming the linearization conditions are valid. An example of such an algorithm is the Linear-Quadratic Regulator (LQR), that uses a constant feedback gain matrix that minimizes the infinite-horizon state error and the quadratic actuator command [52]. Rahmani [38] also developed an optimal reconfiguration maneuver of two spacecraft assuming CWH dynamics. The main conclusion of the work is that a balanced fuel-optimal maneuver of two spacecraft on unperturbed circular orbit is achieved through equal and opposite acceleration of both spacecraft. However, those conclusions do not necessarily apply to elliptical and perturbed orbits. In fact, Inalhan [25] demonstrated that assuming that the reference orbit is circular, even when the eccentricity is as small as 0.005, leads to significant increase of fuel cost because the spacecraft "fights" the natural dynamics to keep the same relative trajectory as it would in a circular orbit.

Therefore, other controllers have been developed for elliptical reference orbits. Such is the case for the Cartesian coordinates feedback control law [46]. If the desired trajectory is described as an inertial position and an inertial velocity, a control feedback law based on Cartesian coordinates errors can be used. Assuming the relative orbits are $J_2$-invariant and that the distance between the spacecraft is small, this simple feedback control law can make use of the non-linear dynamics (such as $J_2$-perturbed dynamics) to compensate position and velocity errors. On the other hand, the mean orbit elements feedback control law, as developed by Schaub et al. [47, 46], uses an error quantified
in terms of relative orbit element errors. It is thus possible to "cooperate" with the physics of orbital dynamics by acting directly on the mean orbit elements to control specific orbit elements at specific instants on the orbit and increase the fuel efficiency of the algorithm. For example, it is much more fuel efficient to correct an inclination error at equator than at the pole, while an error in the ascending node is easier to compensate near the poles. By carefully choosing the time-varying gain matrix of the controller, those effects can be accounted for. Similarly, the hybrid feedback law [46, 45] uses desired states defined as a set of orbit element differences with a reference orbit, while the tracking errors are Cartesian coordinates errors. The main advantage of this method is that the controller uses inputs that are easily measured (relative position and velocity in orbital frame) while the reference is defined as orbit elements, which is more conveniently expressed than rapidly evolving Cartesian coordinates.

However, none of these feedback controllers can ensure optimality for elliptical reference orbits. The presence of several constraints in the problem, the non-linearity of the dynamics and the need for optimality makes the optimal control theory a candidate of choice for formation flying. This theory can fuel-optimize or time-optimize any reconfiguration maneuver while considering perturbed and non-linear dynamics. For circular reference orbits, an analytical solution can be obtained to get an analytical feedback law [33]. However, for reasonably complex dynamics, such as formation flying about an elliptical reference orbit, this method does require highly demanding numerical optimization which could not be implemented on-board. However some near-optimal control methods, like the use of neighbouring optimal paths, only require to solve the optimal maneuver problem for one of the spacecraft. The other spacecraft of the formation can be considered as "neighbours" of this optimal path, and the resulting command offsets can be easily computed. Therefore, the complexity of the problem does not necessarily grow with the number of spacecraft in the formation, as is typically the case with optimal control solutions. Some more recent work [21] demonstrates how such a neighbouring optimal feedback controller can be applied to formation flying. This controller requires very few computation, which facilitates on-board implementation. Furthermore, the formation accuracy/fuel consumption trade-off can easily be
implemented with the selection of only one weight. Moreover, as opposed to the other formation flying feedback controllers, this controller ensures near-optimality for all the elements of the formation.

This controller seeks to minimize second-order variations of the cost function $J$:

$$ J = \frac{1}{2} \left| \Delta e(t_f) \right|^T K \left| \Delta e(t_f) \right| $$

$$ + \int_{t_0}^{t_f} \frac{1}{2} (u)^T R(u) dt $$

(5.5)

where $\Delta e(t_f)$ is the error at final time $t_f$, $u(t)$ is the commanded control effort and $K$ and $R$ are user-defined weighting matrices. The resulting feedback controller [21] takes the form:

$$ u(t) = -H_u^{-1} f_u^T S(t) \Delta e(t) $$

(5.6)

where $H_u$ is a constant diagonal matrix weighting the control effort, $f_u$ is the derivative of the dynamics of the leader with respect to $u$ (which is essentially the matrix form of the Gauss variational equations), $S(t)$ is a pre-computed time-varying 6 x 6 gain matrix and $\Delta e(t)$ is the instantaneous relative orbit elements error. Since both $f_u$ and $S(t)$ are only based on the current value of the uncontrolled leader orbit-element vector, the same gains can be applied to all the elements of the formation guaranteeing near-optimal results for all the elements of the formation. However, near-optimality is guaranteed only at the final time $t_f$. The relative position error of the spacecraft between the initial time $t_0$ and the final time $t_f$ is not minimized, i.e. the controller will accept a large error at time $t$ where $t_0 \leq t < t_f$ if this is to minimize the cost function at the final time $t_f$. Therefore, this controller is oriented toward formation reconfiguration as opposed to formation maintenance, the latter requiring a small position error for all the elements of the formation at every moment. For formation maintenance, other controllers such as the LQR or the mean orbit element controller [47, 46] are better suited since they only consider the current value of the error in their feedback law.

This neighbouring optimum feedback control law will therefore be a part of the autonomous guidance and control loop. It provides a fuel-efficient but nevertheless ac-
accurate way of taking the spacecraft from their initial location to their desired configuration over a user-defined time frame.

5.5 Formation Guidance

Now that all the building blocks are in place, the last step to get a completely autonomous guidance and control system is to build a guidance strategy, using these tools, that is able to autonomously perform trade-offs between formation accuracy, fuel expenditure and equal sharing of the fuel expenditure within the formation.

The purpose of the guidance algorithm is to provide a desired set of relative orbit elements at every moment of the trajectory for all the elements of the formation. The strategy that is proposed here is to command the desired relative orbit elements to naturally bring the spacecraft from its current location to the desired location at the final true anomaly $\nu$. In other words, if the controller was to be perfect and would instantaneously compensate any errors between the current and the desired relative orbit elements, no further control effort should be required to reach the targeted position at the final time. This can be easily done with the state transition matrix presented earlier:

$$\delta e_{\text{des}} = \Phi(e_0, \nu)^{-1} \delta X_{\text{des}}(\nu)$$

(5.7)

This way, the desired states $\delta X_{\text{des}}(\nu)$ of every spacecraft can be mapped back to a current desired position $\delta e_{\text{des}}$. The error to be compensated by the controller is therefore the difference between the current relative orbit elements $\delta e_0$ and the current desired location $\delta e_{\text{des}}$. Such a task can be performed by the neighbouring optimum controller, which will ensure that this error is reduced at the final time $t_f$. Consequently, the main challenge in the design of the guidance algorithm is to identify this desired position $\delta X_{\text{des}}(\nu)$ for every spacecraft.

The proposed guidance algorithm computes the set of desired positions for all spacecraft by minimizing a cost function $J$ that takes into account the total fuel expenditure, the accuracy of the formation and the inequalities between the fuel cost for each spacecraft:
5.5. FORMATION GUIDANCE

Figure 5.1 Projected Circular Formation in Hill Coordinates

\[ J = J_{\text{fuel}} + J_{\text{for}} + J_{\text{diff}} \]  
(5.8)

where \( J_{\text{fuel}} \) is a cost linked to the total fuel expenditure of the formation, \( J_{\text{for}} \) is a cost linked to the accuracy of the formation and \( J_{\text{diff}} \) associates a cost to having differences between the planned fuel cost of the elements of the formation.

For demonstration purposes, this guidance strategy is applied here to the common Projected-Circular Formation (PCF). The PCF is a formation for which all members of the formation are at the same distance from the center of the formation in the normal-tangential plane (Fig. 5.1). As seen from Earth, all members are distributed on a circle. This could have several application for Earth observation.

In Hill coordinates (relative positions \( x, y \) and \( z \) and relative velocities \( V_x, V_y \) and \( V_z \)), the projected circular formation is constrained by:

\[ \rho = \sqrt{y^2 + z^2} \]  
(5.9)
where $\rho$ is the radius of the PCF. The admissible sets of Hill coordinates for a given $\rho$ can be obtained by sweeping the circular formation angular position $\beta$:

\[
\begin{align*}
x(\beta) &= -\frac{\rho}{2} \cos \beta \\
y(\beta) &= \rho \sin \beta \\
z(\beta) &= \rho \cos \beta \\
V_x(\beta) &= \frac{\rho n}{2} \sin \beta \\
V_y(\beta) &= \rho n \cos \beta \\
V_z(\beta) &= \rho n \sin \beta
\end{align*}
\]

where $n$ is the orbital mean motion. The required set of relative orbit elements $\delta e_{pcf}(\beta)$ to reach the corresponding set of Hill coordinates at a desired orbit location can be obtained:

\[
\delta e_{pcf}(\beta) = \Phi^{-1} \delta X(\beta)
\]

where $\delta X(\beta) = \begin{bmatrix} x & y & z & V_x & V_y & V_z \end{bmatrix}^T$. We shall assume here a formation of three spacecraft (excluding the "leader", that could be another spacecraft or a virtual point in space) and that the formation needs to be reached exactly one orbit later. In this context, the guidance problem consists of identifying the angular position vector $\beta = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix}$ containing the targeted angular position for each of the spacecraft that will globally minimize the cost function $J$.

5.5.1 Absolute Fuel Cost Minimization

Obviously, the first objective is to minimize the global amount of propellant that is to be spent to reach the formation. That can be done by minimizing the sum of the predicted velocity impulses of all the spacecraft with:

\[
J_{fuel} = K_{fuel} \sum_{i=1}^{N} \frac{1}{m_i} \Delta V_i(\beta_i)
\]

where $N$ is the total number of spacecraft of the formation (3 in the current example), $m_i$ is the remaining mass of fuel on-board the $i^{th}$ spacecraft and $\Delta V_i(\beta_i)$ is the predicted fuel cost, in the fuel-equivalent space, for the $i^{th}$ spacecraft to reach the angular
position $\beta_i$ (by computing the distance between $\delta e_{pcf}(\beta_i)$ and the initial location of the $i^{th}$ spacecraft $\delta e_{in}$). The remaining mass of propellant $m_i$ has been added as a way of increasing the cost for spacecraft with less fuel remaining. This way, the guidance algorithm will work toward an equal quantity of fuel remaining on-board all spacecraft in the case where not all spacecraft have the same amount of fuel remaining on-board before the maneuver. Finally $K_{fuel}$ is a scalar gain that the user can fit to particular needs depending on how much importance is given to the total fuel expenditure.

5.5.2 Formation Accuracy Optimization

Another important objective of the guidance algorithm is to make sure the spacecraft reach the formation defined by the user. One of the strategies of doing so is by minimizing a potential function based on how well the targeted positions comply with the desired formation. For the present example, the targeted formation consists in placing all three elements of the formation uniformly distributed on the PCF, i.e, with a 120 deg angle between each of the spacecraft as seen from Earth. This desired formation will naturally minimize a potential function based on the sum of the squared angular distances between all the elements of the formation, such as:

$$J_{for} = K_{for} \sum_{i=1}^{N-1} \sum_{n=i+1}^{N} \frac{1}{(\beta_i - \beta_n)^2}$$

(5.18)

where $K_{for}$, similarly to $K_{fuel}$ is a user-defined gain. For other types of formation, $J_{for}$ could be any kind of indicator of the formation accuracy, such as a quality shape factor for a tetrahedral formation [15]. The only requirement is for this value to be small if the proposed configuration is close to the desired formation and large otherwise.

5.5.3 Minimization of the Fuel Cost Difference

Finally, one would want the planned maneuver to minimize as much as possible the difference in the planned fuel cost between all the elements of the formation. The simplest way of doing so is by computing a cost $J_{diff}$ based on the difference between the
maximum and the minimum planned $\Delta V$:

$$J_{fdiff} = K_{fdiff} \left[ \max_i (\Delta V_i) - \min_i (\Delta V_i) \right]$$

(5.19)

where $K_{fdiff}$ is a third gain to be set by the user.

The impact of this cost is not to be confused with the use of the $m_i$ factor in the computation of $J_{fuel}$. The $m_i$ factor is added to compensate spacecraft having different amount of fuel remaining on-board before the maneuver, while the $J_{fdiff}$ term ensures the planned maneuver will not induce a large difference between the fuel consumption of each spacecraft. The term $J_{fdiff}$ can be interpreted as a preventive measure, and the use of $m_i$ as a corrective measure.

### 5.5.4 Optimization Process

Now, with a clearly defined cost function, one has to identify the set of angular positions $\beta$ that optimizes it. Obviously, the optimization process is very mission-specific. Moreover, this optimization process is critical as it is the only part of the guidance and control loop that is not based on analytical solutions. An example of how this could be done with a PCF with very few computation is shown next. However, this optimization process would have to be adapted to each particular mission scenario depending on the parameters that have to be optimized and the number of elements of the formation.

The first step of the optimization is to identify an initial guess. One could start from the set of angular positions that will individually minimize the fuel consumption of each spacecraft. This can be done almost analytically with very few numerical iterations in the fuel-equivalent space (see Ref. [18]). However, the danger of using a fixed initial guess is that the search process could remain trapped in a local minimum, therefore missing the global minimum. One efficient way of avoiding local minima in this case consists in performing a global coarse search of all the possible solutions. Sampling the complete solution space with 0.5 rad wide steps generates 2197 points where the cost function needs to be computed (13 positions for the three spacecraft), which is
very reasonable since the evaluation of the cost function is based on simple analytical expressions. The initial location for the refinement process would be the vector out of the 2197 with the lowest cost function value.

A refinement process is started from this initial location. Each of the three angular positions is moved by a smaller step \( d\beta \) forward then backward (Fig. 5.2). This leaves us with the evaluation of six new cost function values (plus or minus 0.01 rad for each of the three spacecraft in this particular case). If any of these new configurations lower the cost function value, then the configuration (out of the 6) with the lowest cost function is used as the baseline. The process is then repeated until the cost function reaches a minimum. On all cases simulated, this optimization process always took less than 1 sec on a common desktop computer.

5.6 Simulation Examples

Finally, an example is given here of how this can be applied to a LEO PCF. The problem consists in bringing three spacecraft initiated at arbitrary locations to an equally distributed 1 km size PCF, one orbit later, while minimizing fuel expenditure and the difference in fuel consumption between all spacecraft.
CHAPTER 5. CLOSING THE LOOP

TABLE 5.1 Reference Initial Orbit Elements

<table>
<thead>
<tr>
<th>$e_0$</th>
<th>$a_0$</th>
<th>1.1$R_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_0$</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>$i_0$</td>
<td>$\pi/4$</td>
<td></td>
</tr>
<tr>
<td>$\Omega_0$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\omega_0$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$M_0$</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

TABLE 5.2 Spacecraft Initial Orbit Element Offset

<table>
<thead>
<tr>
<th>S/C 1</th>
<th>S/C 2</th>
<th>S/C 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta a_0$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\delta e_0$</td>
<td>+0.0001</td>
<td>+0.0001</td>
</tr>
<tr>
<td>$\delta i_0$</td>
<td>+0.0001</td>
<td>+0.0001</td>
</tr>
<tr>
<td>$\delta \Omega_0$</td>
<td>+0.0001</td>
<td>-0.0001</td>
</tr>
<tr>
<td>$\delta \omega_0$</td>
<td>-0.0001</td>
<td>-0.0001</td>
</tr>
<tr>
<td>$\delta M_0$</td>
<td>+0.0001</td>
<td>+0.0001</td>
</tr>
</tbody>
</table>

Table 5.1 presents the initial orbit elements of the chief, i.e. the semimajor axis $a$, the eccentricity $e$, the inclination $i$, the right ascension of the ascending node $\Omega$, the argument of perigee $\omega$ and finally the mean anomaly $M$. Table 5.2 shows the initial relative orbit elements of the three other spacecraft. As a reference, the resulting $J_2$-perturbed uncontrolled relative motion of all three spacecraft with respect to the leader are shown in Fig. 5.3. On this figure, trajectories are also plotted with respect to the targeted 1 km size PCF (dotted circle).

The first simulation scenario considers only formation accuracy by setting $K_{fuel} = 0$, $K_{for} = 1$ and $K_{diff} = 0$. Figure 5.4 shows the resulting out-of-plane relative trajectory of the three spacecraft (as seen from Earth), while Fig. 5.5 presents the time history of the total cumulated fuel cost for each of the spacecraft. As expected, the guidance algorithm commands a uniform distribution of the spacecraft (all the spacecraft are
The second simulation scenario considers fuel cost as the only criterion to optimize the function by setting $K_{fuel} = 1$, $K_{for} = 0$ and $K_{diff} = 0$. In this case, all the spacecraft are forced to end up on a 1 km size PCF, but the selection of the angular location of every spacecraft is only based on the minimization of the fuel consumption. No importance is given to the distribution of the spacecraft on the PCF. Figure 5.6 shows the resulting out-of-plane relative trajectory of the three spacecraft and Fig. 5.7 presents the time history of the total cumulated velocity impulse for each of the spacecraft. As can be noted in Fig. 5.6, considering only fuel cost does not lead to a well-balanced formation. In fact, spacecraft 1 and 3 practically end up at the same location. However, fuel cost is much lower than if only the formation accuracy is considered.

The third simulation scenario consists in reaching the PCF only by trying to minimize the fuel difference between all spacecraft (Fig. 5.8 and Fig. 5.9). In this case, the fuel expenditure is well balanced between spacecraft (Fig. 5.9), but the formation is not well balanced (Fig. 5.8).
Figure 5.4 Resulting trajectory if only formation cost is considered. Circles mark initial locations of spacecraft. Diamonds mark final locations. X's mark targeted locations as computed by the guidance algorithm.

Figure 5.5 Cumulated fuel expenditure of each spacecraft if only formation accuracy is considered in the cost function.
5.6. SIMULATION EXAMPLES

Figure 5.6 Resulting trajectory if only total fuel expenditure is considered. Circles mark initial locations of spacecraft. Diamonds mark final locations. X's mark targeted locations as computed by the guidance algorithm.

Figure 5.7 Cumulated fuel expenditure of each spacecraft if only total fuel expenditure is considered in the cost function.
Figure 5.8 Resulting trajectory if only differential fuel expenditure is considered. Circles mark initial locations of spacecraft. Diamonds mark final locations. X’s mark targeted locations as computed by the guidance algorithm.

Figure 5.9 Cumulated fuel expenditure of each spacecraft if only differential fuel expenditure is considered in the cost function.
Figure 5.10 Resulting trajectory if all terms of cost function are considered. Circles mark initial locations of spacecraft. Diamonds mark final locations. X's mark targeted locations as computed by the guidance scheme.

Finally, Fig. 5.10 and Fig. 5.11 show the resulting trajectory and fuel cost if all 3 terms of the cost function are used, \( i.e. K_{fuel} = 0.1, K_{for} = 1, K_{diff} = 0.5 \). This shows a case where the obtained formation (Fig. 5.10) is close to the desired formation (Fig. 5.4) but with a much lower total fuel cost (Fig. 5.11 and Fig. 5.5).

Obviously any other trade-off can be achieved through the selection of the relative values of \( K_{fuel}, K_{for} \) and \( K_{diff} \). As the simulation results presented here have shown, these gains have a lot of influence on the final configuration and the fuel spent by each spacecraft to reach the formation.

### 5.7 Conclusion

This paper presented how an autonomous formation flying guidance and control loop can be established. This paper first summarized the design of three tools: a linearized relative motion model for \( J_2 \)-perturbed eccentric orbits, the fuel-equivalent space and the neighbouring optimum controller, which are the building blocks of the system. Then, a guidance algorithm which autonomously performs trade-offs between fuel
In Figure 5.11 Cumulated fuel expenditure of each spacecraft if all terms are considered in the cost function.

consumption, formation accuracy and equality of the fuel expenditure was developed. Simulation results have shown that this guidance algorithm, linked with the neighbouring optimum controller, is a very efficient way of autonomously performing maneuvers on-orbit over a 1 orbit time-frame. Such a demonstration has been completed with a typical 1 km size LEO PCF.

Even though this algorithm is applied here to the PCF, it can be adapted to other types of formation, such as the tetrahedral formation. The only requirement is to have an easy way of measuring the quality of the formation commanded by the guidance system.

For other types of formation, the optimization parameters and the optimization process would have to be different. Even for a PCF, one could think of several other parameters that could be optimized, such as location of the leader (or the virtual center) or the size of the formation. However, limiting the number of optimization parameters drastically reduces the computational power required (especially for a large formation) which largely facilitates implementation on a computational power-limited on-board computer.
Further developments of such a system could include collision avoidance and decentralization of the decision process. Indeed, as of now, only the initial position and the final location of the spacecraft are considered. What happens in-between is left to the controller, which is not yet adapted to include collision-avoidance in the planning process. Furthermore, the guidance algorithm presented here uses a centralized approach, i.e. the algorithm has all the information regarding all the spacecraft to make its decision. The decentralization of this algorithm could make it more robust to spacecraft failure, since no single point would be responsible to make decisions for the whole fleet. This could be accomplished by designing individual cost functions for each of the spacecraft that require a minimal information exchange between the spacecraft.
PART IV

Conclusion and Future Work
CHAPTER 6

Conclusion

This research project focused on the development of solutions for autonomous forma-
tion flying guidance and control. As was presented in the review of the literature, sev­
eral key challenges still needed to be overcome before achieving such an autonomous
system. Firstly, it has been shown that an analytical model of relative motion that takes
into account the eccentricity of the reference orbit and the effects of the $J_2$ perturbation
suitable for autonomous on-board applications was still to be developed. Few ana­
lytical relative motion models include both the effect of the perturbation caused by $J_2$
and the effect of orbit eccentricity. The only two that do so, the relative mean orbit
element propagation and the Gim-Alfriend state transition matrix, both require prop­
agation of the states of the reference trajectory to perform the mean to osculating and
osculating to mean mapping (even though they are in an analytical form). On the other
hand, most of the work on formation flying guidance uses highly demanding numeri­
cal optimization and is designed to run off-line on ground-based powerful computers
as opposed to limited-capacity on-board computers. Few analytical solutions to the
reconfiguration exist and these solutions are mostly applicable to specific formations
and circular reference orbits. Moreover, there exists several analytical solutions to the
relative motion control problem, but none of the proposed controllers can guarantee
the fuel-optimality of the maneuver within a reasonable time frame. The performance
of these controllers essentially relies on a proper tuning of the controller gains, which
is very difficult to perform.

Based on these observations, three project objectives were defined: (1) the development
of an analytical model of relative motion for $J_2$-perturbed elliptical reference orbits, (2)
the development of analytical tools to perform formation flying guidance about ellip­
tical reference orbits and finally (3) the development of an optimal or near-optimal
formation flying relative motion feedback control strategy suitable for on-board applications.

The first objective was achieved with the development of a model of linearized relative motion about $J_2$-perturbed elliptical orbits (Chapter 2). This model accurately models the impact of secular drift caused by the $J_2$ perturbation on the osculating relative motion, without the need to propagate the states of the reference trajectory forward in time. This can be done at the cost of neglecting the impact of short-periodic terms on relative motion, which inevitably leads to a bounded prediction error. As it has been shown, this residual error does not grow with time which means that its accuracy remains the same even for long-term relative motion prediction, assuming only the $J_2$ perturbation is present. Only the impact of the other neglected perturbations (solar radiation pressure, drag, other gravitational harmonics, etc.) will affect the long-term prediction accuracy of the model.

The second objective was achieved with the development of the Fuel-Equivalent Space (Chapter 3). This new mathematical tool translates the relative orbit elements into a Fuel-Equivalent Space where a similar displacement on any axis is identical in terms of fuel cost. It makes use of the Impulsive Feedback Controller theory, which was originally designed to perform small orbit corrections with only three impulses while minimizing the coupling between the correction of each orbit element. Once the initial relative location of the spacecraft and the targeted formation have been mapped into the Fuel-Equivalent Space, the minimum-fuel reconfiguration problem becomes a simple geometric distance minimization problem in the Fuel-Equivalent Space. Two such examples were given: the $J_2$-invariant relative orbits and the Projected-Circular Formation. For both cases, it is shown that the mapping of the problem in the Fuel-Equivalent Space leads to simple analytical solutions or to solutions that can be found with few and simple iterations.

The third objective was achieved with the application of the neighbouring-optimum control theory to the formation flying problem (Chapter 4). The neighbouring-optimum feedback law, based on optimal control theory, can perform the fuel/formation accu-
racy trade-off with the selection of the value of only one scalar gain. The controller is in the semi-analytic form, as only one gain matrix needs to be pre-computed for the complete trajectory. However, once the time-history of this matrix is known, the controller guarantees near-optimal (in terms of fuel and formation accuracy) maneuvering for all the spacecraft evolving in the vicinity of the uncontrolled reference trajectory. Simulation results have shown that this controller can perform a maneuver with less fuel and with errors several orders of magnitude smaller than other common formation flying controllers, namely the LQR and the mean orbit elements controller. Simulation results have also shown that, as theory predicted, the control signal computed by the neighbouring-optimum feedback law essentially recreates the optimum open-loop command, but in a feedback form.

Finally these three new developments were tied in together to form a fully autonomous guidance and control loop. These tools are connected together with a guidance algorithm based on a cost function that takes into account formation accuracy, minimization of fuel and balancing of fuel spending among the formation. Once the desired formation and location is specified, the system autonomously selects the best location on the formation for each spacecraft and controls it toward this location with a near-optimal fuel spending. It is also shown that with the relative selection of the formation, fuel-spending and fuel-balancing weights (each represented by one scalar gain), the user can easily perform the trade-off between these requirements.

6.1 Summary of the Contributions

The outcome of the research project can be classified into five main contributions (Fig. 6.1). The first one is an identification of the missing links in the formation flying literature in order to get a fully autonomous guidance and control loop [16], as presented at the International Aeronautical Congress in October 2006. Then, the analytical relative motion model for perturbed elliptical reference orbits [19, 20], the Fuel-Equivalent Space theory [18, 22] and the neighbouring optimum feedback law [21] were presented to the AAS/AIAA Astrodynamics Specialist Conference in August 2007 and submitted...
to the AIAA Journal of Guidance, Control, and Dynamics (two out of three submissions have been accepted at the time of writing this document). Finally the complete guidance and control loop [17] was presented at the International Aeronautical Congress in September 2007. In summary, the candidate research work led to five international conference papers, 2 accepted journal submission and 1 pending journal submission.

6.2 Future Work

Future work on this topic could firstly include the possibility to perform collision avoidance. The proposed guidance and control system globally minimizes a cost function for the whole formation at the beginning of the maneuver, but does not consider the relative position of the spacecraft during the maneuver. Each spacecraft is controlled independently once the maneuver is planned. The inclusion of potential functions, for example, in the control sequence could be a way of taking into account collision avoidance during the maneuvers. Also, the proposed guidance algorithm only uses the cost of maneuvering to perform its computations. However, formation maintenance is also a major component of the fuel cost of a mission. In the case where more than a “snap-shot” formation is needed, i.e. if the formation is to be maintained, the cost of maintaining the assigned location of each spacecraft could also be included in the guidance cost function. Finally, one could think of decentralizing the system. Currently, one “omniscient” member of the formation performs the planning for the whole formation. Even though it has been shown that the planning is not demanding in terms of computing power, a decentralized approach should be more robust to spacecraft failure. The impact on the performance of the system of the consequent minimization of the exchange of information between the spacecraft could be assessed.
Figure 6.1 Research Project Contributions. Candidate contributions and publications are represented by dark-shaded boxes.
PART V

Appendices
This section presents a summary of the definition of the classical orbit elements used throughout this work.

**Semimajor Axis** \((a)\) Major axis length of the orbital ellipse \((\text{m})\) (Fig. A.1). Defines the orbital energy and the orbital period.

**Eccentricity** \((e)\) Measure of how much the orbit deviates from a circle (Fig. A.1).

**Inclination** \((i)\) Angle between the orbital plane and the equatorial plane \((\text{rad})\) (Fig. A.2).

**Right Ascension of the Ascending Node** \((\Omega)\) Angle, in the equatorial plane, between the ascending node (where the orbit crosses the equatorial plane from South to North) and the reference direction, typically the First Point of Aries for Earth-orbiting bodies \((\text{rad})\) (Fig. A.2).

**Argument of Periapsis** \((\omega)\) Angle, in the orbital plane, between the periapsis (point on the orbit that is the closest from the central body) and the equatorial plane \((\text{rad})\) (Fig. A.1 and A.2). Also referred to as “argument of perigee” for Earth-orbiting bodies.

**True Anomaly** \((v)\) Angle, in the orbital plane, between the position of the spacecraft and the line of periapsis \((\text{rad})\) (Fig. A.1).

**Eccentric Anomaly** \((E)\) Angle between the direction of periapsis and the current position of the spacecraft on its orbit, projected onto the orbit ellipse’s circumscribing circle perpendicularly to the major axis, and measured at the center of the ellipse \((\text{rad})\) (Fig. A.3).

**Mean Anomaly** \((M)\) Measure of the fraction of the orbit period (expressed as an angle) that the spacecraft has moved since its last passage at periapsis \((\text{rad})\) (Fig. A.3). The relationship between the eccentric anomaly and the mean anomaly is \(M = E - e \sin E\).

**Argument of Latitude** \((\theta)\) Angle, in the orbital plane, between the spacecraft and the equatorial plane \((\text{rad})\) (Fig. A.2). The argument of latitude is the sum of the true anomaly and the argument of perigee \((\theta = \omega + v)\).
Figure A.1 Definition of the Orbit Elements in the Orbital Plane.

Figure A.2 Orbital Plane and Equatorial Plane.
Figure A.3 Representation of the True, Mean and Eccentric Anomalies.
BIBLIOGRAPHY


